Day 280 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Compound change with logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A colony of 500 cells grows by 8% each day.
Find the least whole number of days for the colony to exceed 2500 cells.

$500 \times 1.08^n > 2500$, so $1.08^n > 5$

$n > \frac{\ln 5}{\ln 1.08} = 20.91$

n must be a whole number, so n = 21

Answer: 21

2 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ab^x$. The line of best fit for lg y against x is lg y = 0.4 + 0.6x.
(a) Find a and b to 3 significant figures.
(b) Estimate y when x = 14, to 3 significant figures.
(c) Interpret the value of b.

$y = ab^x$ gives $\lg y = \lg a + x \lg b$

$\lg a = 0.4$ so $a = 10^{0.4} = 2.512$

$\lg b = 0.6$ so $b = 10^{0.6} = 3.981$

x = 14: $\lg y = 0.4 + 0.6 \times 14 = 8.8$

$y = 10^{8.8} = 631000000$

b is the factor y is multiplied by each time x increases by 1 (growth of 300%)

Answer: (a) a = 2.51, b = 3.98 (b) 631000000 (c) y grows by a factor of 3.98 per unit of x

3 Differentiating a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 5^x - 2x$ has one stationary point.
Find its coordinates to 3 s.f. and show that it is a minimum.

$dy/dx = 5^x \ln 5 - 2 = 0$

$5^x = \frac{2}{\ln 5}$, $x = \frac{1}{\ln 5} \ln(\frac{2}{\ln 5}) = 0.135$

$d^2y/dx^2 = 5^x(\ln 5)^2 > 0$ for all x: minimum

$y = \frac{2}{\ln 5} - 2 \times x = 0.973$

Answer: (0.135, 0.973)

4 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 2x^3 + ax^2 + bx + 16$, where a and b are constants.
(x + 1) is a factor of f(x), and f(x) leaves remainder 144 when divided by (x − 2).
Find a and b, and hence solve f(x) = 0.

f(−1) = 0: −2 + a − b + 16 = 0, so a − b + 14 = 0.

f(2) = 144: 4a + 2b + 32 = 144.

Solving: a = 14, b = 28.

$f(x) = 2x^3 + 14x^2 + 28x + 16 = 2(x + 4)(x + 2)(x + 1)$

Answer: a = 14, b = 28; x = −4, −2, −1

5 Segments: exact area and perimeter (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A chord AB of a circle, centre O and radius 3 cm, subtends an angle of $\frac{\pi}{2}$ at O.

Find the exact area and the exact perimeter of the minor segment cut off by AB.

Sector OAB: $\frac{1}{2}r^2\theta = \frac{1}{2} \times 3^2 \times \frac{\pi}{2}$; triangle OAB: $\frac{1}{2}r^2\sin\theta = \frac{9}{2}$

Area = $\frac{9}{4}\pi - \frac{9}{2}$

Chord AB = $2r\sin(\theta/2) = 3\sqrt{2}$

Arc AB = $r\theta = \frac{3}{2}\pi$

Perimeter = arc + chord

Answer: Area = $-\frac{9}{2} + \frac{9}{4}\pi$ cm^2, perimeter = $\frac{3}{2}\pi + 3\sqrt{2}$ cm

6 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each component is faulty with probability 0.1, independently.

Find the least number n of components that must be tested so that the probability of finding at least one faulty is greater than 0.99.

P(at least one faulty) = $1 - 0.9^n > 0.99$

$0.9^n < 0.01$

$n \ln 0.9 < \ln 0.01$

Dividing by ln 0.9 (negative) reverses the inequality: $n > 43.71$

$n = 44$

Answer: n = 44

7 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Evaluate $\sum_{r=5}^{13}(3 \times 2^r + 1r)$.

Geometric part: first term $3 \times 2^5 = 96$, ratio 2, 9 terms.

Sum = $\frac{96(2^9 - 1)}{1} = 49056$.

Arithmetic part: 1×9 terms, average $(5 + 13)/2$: sum = 81.

Total = 49056 + 81.

Answer: 49137

8 Sinusoidal velocity (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. At time t seconds its velocity is v m s^{-1}, where $v = 6\cos(\frac{\pi t}{6})$ for $t \geq 0$.

Find (a) the greatest magnitude of the acceleration of P, (b) the distance P travels before it first comes to instantaneous rest. Give (b) to 3 s.f.

$a = \frac{dv}{dt} = -1\pi\sin(\frac{\pi t}{6})$, so the greatest magnitude is 1π

$v = 0$ when $\cos(\frac{\pi t}{6}) = 0$: first at $\frac{\pi t}{6} = \frac{\pi}{2}$, so $t = 3$

$s = \int$ from 0 to 3 of $6\cos(\frac{\pi t}{6})\,dt = [\frac{6 \times 6}{\pi}\sin(\frac{\pi t}{6})]$ from 0 to 3

$= \frac{36}{\pi} \times (1 - 0)$

Answer: (a) 1π m s^{-2} (b) 11.5 m

9 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $8y = 6x + k$ is a tangent to the circle
$(x + 0)^2 + (y + 2)^2 = 16$.
Find the two possible values of k.

Substitute $y = \frac{6x + k}{8}$: $(x + 0)^2 + (\frac{6x + k}{8} + 2)^2 = 16$.

Multiply by 64 and collect: a quadratic in x with a = 100.

Tangent ⇒ discriminant = 0. This simplifies to $(k + 16)^2 = 1600$.

$k + 16 = \pm 40$.

Answer: $k = -56$ or $k = 24$

10 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 2 kg lies on a smooth horizontal table, with a block of mass 6 kg on it ($\mu = 0.2$ between them). A horizontal force P N acts on the block.
(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 117.6.
Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{8}$. The plank is accelerated only by friction: $F = 2a$

Limiting friction: $F = \mu mg = 0.2 \times 6 \times 9.8 = 11.76$, so $a = \frac{11.76}{2} = 5.88$

P_max = 8 × 5.88 = 47.04

P = 117.6 > 47.04, so the block slips. Plank: $a = \frac{11.76}{2} = 5.88$

Block: 117.6 − 11.76 = 6 a, so a = 17.64

Answer: (a) 47.04 N (b) block 17.64 m s^{-2}, plank 5.88 m s^{-2}