Day 28 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Equidistant point and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A(−7, −12) and B(−14, 5). The point C lies on the x-axis and CA = CB. Find the coordinates of C and the area of triangle ABC.

C = (c, 0): $(c - (-7))^2 + 144 = (c - (-14))^2 + 25$.
Expand and solve: $-14c = 28$, so $c = -2$.
Area = $\frac{1}{2}|(x_A - c)y_B - (x_B - c)y_A|$
$= \frac{1}{2}|(-5) \times 5 - (-12) \times (-12)| = \frac{169}{2}$.

Answer: C(−2, 0); area $\frac{169}{2}$

2 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The fill volume of a bottle is N(μ, 25) with known variance 25. The mean was claimed to be 91 ml.
A random sample of 16 has mean 92. Test at the 10% level whether the mean is different from 91.

H_0: μ = 91, H_1: μ ≠ 91. Under H_0, the sample mean ~ N(91, $\frac{25}{16}$)

$z = \frac{92 - 91}{5 \div \sqrt{16}} = 0.800$
One tail: P(Z > 0.800) = 0.212, compared with 5%
0.212 > 5%, so do not reject H_0
There is insufficient evidence at the 10% level that the mean has changed as claimed
Answer: Do not reject H_0 (z = 0.800, p-value 0.212)

3 Projectile from a cliff (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from the top of a vertical cliff, 88.2 m high, at 24.5 m s^{-1} at angle α above the horizontal ($\tan \alpha = \frac{3}{4}$), and lands on horizontal ground.
Find (a) the time of flight and the distance from the cliff, (b) the speed and direction on landing. Take g = 9.8 m s^{-2}.

Vertically (up positive): $-88.2 = 14.7t - 4.9t^2$, so $4.9t^2 - 14.7t - 88.2 = 0$
t = 6 s (the positive root)
Horizontal distance = 19.6 × 6 = 117.6 m
On landing: $v_x = 19.6$, $v_y = 14.7 - 9.8 \times 6 = -44.1$
speed = $\sqrt{(19.6^2 + 44.1^2)} = 48.3$, tan angle = $\frac{44.1}{19.6}$, angle = 66.0° below the horizontal

Answer: (a) 6 s, 117.6 m (b) 48.3 m s^{-1} at 66.0° below the horizontal

4 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $(n + 2)^3 - n^3 - 8$ is a multiple of 6 for every integer n.

$(n + 2)^3 = n^3 + 6n^2 + 12n + 8$.
So the expression = $6n^2 + 12n$
$= 6n(n + 2)$, a multiple of 6.
Answer: Proved

5 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{2}{17 + 8\sin\theta + 6\cos\theta}$

(a) Write $8\sin\theta + 6\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{8^2 + 6^2} = 10$, $\tan\alpha = \frac{6}{8}$, $\alpha = 36.9°$

Denominator $= 17 + 10\sin(\theta + 36.9°)$, which lies between 7 and 27

Greatest $f(\theta) = \frac{2}{7} = \frac{2}{7}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{2}{27} = \frac{2}{27}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 36.9° = 270°$, so $\theta = 233.1°$

Answer: (a) $10\sin(\theta + 36.9°)$ (b) greatest $\frac{2}{7}$, least $\frac{2}{27}$, $\theta = 233.1°$

6 Coded regression line (AQA, Edexcel, OCR, MEI)

Data pairs (x, y) are coded by $u = x - 6$ and $v = \frac{y}{5}$. The regression line of v on u is $v = 4.4 + 1.8u$.

(a) Find the regression line of y on x in the form $y = a + bx$.

(b) Use it to estimate y when $x = 13$.

Substitute $u = x - 6$ and $v = y/5$: $\frac{y}{5} = 4.4 + 1.8(x - 6)$

$y = 5[4.4 + 1.8x - 10.8] = 5 \times -6.4 + 5 \times 1.8x$

$y = -32 + 9x$

At $x = 13$: $y = -32 + 9 \times 13 = 85$

Answer: (a) $y = -32 + 9x$ (b) 85

7 Derivative of the inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve is given by $x = y^3 + y^2 + 5y$.

Find dy/dx and d^2y/dx^2 when $y = -2$.

$dx/dy = 3y^2 + 2y + 5 = 13$ when $y = -2$, so $dy/dx = \frac{1}{dx/dy} = \frac{1}{13}$

$d^2x/dy^2 = -10$ when $y = -2$

$d^2y/dx^2 = -\frac{d^2x}{dy^2/(dx/dy)^3} = -\frac{-10}{13^3}$

Answer: $dy/dx = \frac{1}{13}$, $d^2y/dx^2 = \frac{10}{2197}$

8 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$\frac{dy}{dx} = -4xy$, and $y = 2$ when $x = 0$.

Solve the differential equation and find y when $x = 1$.

$\int \frac{1}{y}\,dy = \int -4x\,dx$, so $\ln y = -4x^2/2 + c = -2x^2 + c$

$x = 0$, $y = 2$: $c = \ln 2$, so $y = 2\,e^{-2x^2}$

$x = 1$: exponent $= -2$

Answer: $y = 2\,e^{-2x^2}$; $y = 2e^{-2}$

9 Sums of multiples (arithmetic series) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the sum of all the integers from 1 to 740
inclusive that are divisible by 6 or by 10, but not by both.

Multiples of 6: 123 terms (6 to 738): sum = 45756.

Multiples of 10: 74 terms: sum = 27750.

Multiples of both, i.e. of 30: 24 terms: sum = 9000.

Required sum $= 45756 + 27750 - 2 \times 9000$.

Answer: 55506

10 Least force to drag a box (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 40 N is pulled at constant speed along rough horizontal ground ($\mu = 0.75$) by a force P N at angle θ above the horizontal.

(a) Find P when $\tan\theta = \frac{4}{3}$. (b) Find the least possible value of P as θ varies, and the corresponding θ to 1 d.p.

Vertical: $R + P\sin\theta = 40$. Horizontal: $P\cos\theta = \mu R = 0.75(40 - P\sin\theta)$
$P(\cos\theta + 0.75\sin\theta) = 0.75 \times 40$, so $P = 30 \div (\cos\theta + 0.75\sin\theta)$

(a) $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$: $P = 30 \div 1.2 = 25$

(b) P is least when $\cos\theta + 0.75\sin\theta$ is greatest. Its greatest value is $\sqrt{(1 + 0.75^2)} = 1.25$, which occurs when $\tan\theta = 0.75$. Then $P = 30 \div 1.25 = 24$ and $\theta = 36.9°$

Answer: (a) 25 N (b) 24 N at $\theta = 36.9°$