Day 277 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ab^x$ passes through (1, 16) and (5, 4096).
Find the exact values of a and b, and the value of y when x = 7.

$ab^1 = 16$ and $ab^5 = 4096$
Divide: $b^4 = 256$ so $b = 4$
$a = 16 \div 4^1 = 4$
$y = 4 \times 4^7$
Answer: a = 4, b = 4, y = 65536

2 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A tennis ball is projected from a point O on horizontal ground with speed 49 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{3}{4}$. Find (a) the greatest height, (b) the speed when it is at half of this height (3 s.f.). Take g = 9.8 m s^{-2}.

Vertical: $0 = 29.4^2 - 2 \times 9.8 \times H$, so $H = 44.1$
At height $\frac{H}{2} = 22.05$: $v_y^2 = 29.4^2 - 2 \times 9.8 \times 22.05 = 432.18$
$v_x = 39.2$ (constant), so speed$^2 = 1536.64 + 432.18 = 1968.82$
Answer: (a) 44.1 m (b) 44.4 m s^{-1}

3 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + x^2 + 7x - 9$.
Show that $(x - 1)$ is a factor of f(x), and that the equation f(x) = 0 has exactly one real root.

$f(1) = 1 + 1 + 7 - 9 = 0$, so $(x - 1)$ is a factor.
$f(x) = (x - 1)(x^2 + 2x + 9)$.
For $x^2 + 2x + 9 = 0$: $b^2 - 4ac = 4 - 36 = -32 < 0$.
No real roots from the quadratic, so the only real root is x = 1.
Answer: Proved; the only real root is x = 1

4 Projectile from a cliff (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected horizontally at 39.2 m s^{-1} from the top of a vertical cliff of height 19.6 m, and lands on horizontal ground.
Find (a) the time of flight and the distance from the cliff, (b) the speed and direction on landing. Take g = 9.8 m s^{-2}.

Vertically (up positive): $-19.6 = -4.9t^2$, so $t^2 = 4$
t = 2 s (the positive root)
Horizontal distance = 39.2 × 2 = 78.4 m
On landing: $v_x = 39.2$, $v_y = 0 - 9.8 \times 2 = -19.6$
speed = $\sqrt{(39.2^2 + 19.6^2)} = 43.8$, tan angle = $\frac{19.6}{39.2}$, angle = 26.6° below the horizontal
Answer: (a) 2 s, 78.4 m (b) 43.8 m s^{-1} at 26.6° below the horizontal

5 One-tailed binomial test with data (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A company claims that 60% of its customers renew their policy. A sample of 15 customers is taken, and 7 renew.
Test, at the 5% significance level, the claim that the proportion is lower.
Give your conclusion in context.

H_0: p = 0.6, H_1: p < 0.6. Let X ~ B(15, 0.6)
$P(X \leq 7) = 0.213$
0.213 is greater than 5%, so do not reject H_0
There is insufficient evidence that the proportion is lower than the claimed 0.6
Answer: Do not reject H_0: p-value 0.213 > 5%

6 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = x^3 + 2x - 10$.
(a) Show that $f(x) = 0$ has a root α with $1 < \alpha < 2$.
(b) Use Newton-Raphson with $x_0 = 1$ to find x_1 and x_2 to 4 decimal places.

$f(1) = -7$ and $f(2) = 2$: a change of sign, f continuous

$f'(x) = 3x^2 + 2$, so $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_1 = 1 - \dfrac{-7.000}{5.000} = 2.4000$

$x_2 = 2.4000 - \dfrac{8.624}{19.28} = 1.9527$

Answer: $x_1 = 2.4000$, $x_2 = 1.9527$

7 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sum_{r=1}^{n}(7r - 2) = 1161$, find n.

$\sum_{r=1}^{n}(7r - 2) = 7 \times \dfrac{n(n + 1)}{2} - 2n.$

$\dfrac{7\,n(n + 1)}{2} - 2n = 1161$, so $7n^2 + 3n - 2322 = 0$.

Solving ($n > 0$): $n = 18$.

Answer: $n = 18$

8 Transformations: new equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 2x^2 + 3x - 6$ is translated by $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ and then a reflection in the y-axis.
Find the equation of the resulting curve in the form $y = ax^2 + bx + c$.

Translation by $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$: $y = f(x + 1) - 2 = 2x^2 + 7x - 3$.

Then a reflection in the y-axis: $x \to -x$.
Expand and simplify to give the quadratic.

Answer: $y = 2x^2 - 7x - 3$

9 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates $(-2, 0, 2)$, $(-1, -1, 4)$, $(5, 4, 2)$.
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$\mathrm{BA} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$, $\mathrm{BC} = \begin{pmatrix} 6 \\ 5 \\ -2 \end{pmatrix}$; $\mathrm{BA} \cdot \mathrm{BC} = 3$

$|\mathrm{BA}| = \sqrt{6}$, $|\mathrm{BC}| = \sqrt{65}$

$\cos \mathrm{ABC} = \dfrac{3}{\sqrt{6}\sqrt{65}} = 0.1519$, so $\mathrm{ABC} = 81.3°$

$|\mathrm{BA} \times \mathrm{BC}| = \left|\begin{pmatrix} 8 \\ -14 \\ -11 \end{pmatrix}\right| = \sqrt{381}$

$\text{Area} = \dfrac{1}{2}|\mathrm{BA} \times \mathrm{BC}|$

Answer: ABC = 81.3°, area = $\dfrac{1}{2}\sqrt{381}$

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10 Conditional probability with a normal variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bulb's lifetime, in hours, is N(86, 8^2).
Given that it exceeds 94, find the probability that it exceeds 104.

Required = $P(X > 104 \mid X > 94) = \frac{P(X > 104)}{P(X > 94)}$

$P(X > 104) = P(Z > 2.250) = 0.01222$

$P(X > 94) = P(Z > 1.000) = 0.1587$

$= \frac{0.01222}{0.1587} = 0.0771$

Answer: 0.0771