Day 276 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Compound change with logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A van costs £24000 and loses 8% of its value each year.
Find the least whole number of years after which its value is below £5000.

$24000 \times 0.92^n < 5000$, so $0.92^n < \frac{5}{24}$

$n > \frac{\ln \frac{5}{24}}{\ln 0.92} = 18.81$ (ln 0.92 is negative, so the inequality reverses)
$n = 19$
Answer: 19

2 Transformations of y = x^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Describe a sequence of two transformations that maps the curve $y = x^2$ onto the curve $y = 3x^2 - 12x + 17$.

Complete the square: $y = 3(x - 2)^2 + 5$.
Step 1: $y = x^2 \rightarrow y = 3x^2$: a stretch, scale factor 3, parallel to the y-axis.
Step 2: replace x by $(x - 2)$ and add 5: translation by $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.
Answer: a stretch, scale factor 3, parallel to the y-axis, then translation $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

3 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (2.5t^2 + 2t)\mathbf{i} + (3t^2 - 4t)\mathbf{j}$.
Find (a) the value of t at which P is moving parallel to $(\mathbf{i} + \mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (5t + 2)\mathbf{i} + (6t - 4)\mathbf{j}$
Parallel to $(\mathbf{i} + \mathbf{j})$: $(5t + 2) \times 1 = (6t - 4) \times 1$
$t(-1) = -6$, so $t = 6$
At $t = 6$: $\mathbf{v} = (32\mathbf{i} + 32\mathbf{j})$, speed $= \sqrt{(32^2 + 32^2)}$
Answer: (a) t = 6 (b) 45.3 m s^{-1}

4 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(26, \frac{1}{4})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{26 - r}{r + 1} \times \frac{1}{3}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(26, r + 1)}{C(26, r)} \times \frac{p}{q} = \frac{26 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{1}{4}$ and $q = \frac{3}{4}$, $p \div q = \frac{1}{3}$

The ratio exceeds 1 when $(26 - r)\frac{1}{4} > (r + 1)\frac{3}{4}$, that is $r < 5.75$

So probabilities rise up to r = 6 and then fall; the mode is 6
$P(X = 6) = 0.178$
Answer: 6 with probability 0.178

5 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 20 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 80$, and after 4 minutes $\theta = 62$.
Find A and k, then find when $\theta = 27$ (3 s.f.).

$t = 0$: $20 + A = 80$ so $A = 60$

$t = 4$: $20 + 60e^{-4k} = 62$ so $e^{-4k} = \frac{7}{10}$

$k = \frac{1}{4} \ln \frac{10}{7} = 0.0892$

$60\, e^{-kt} = 7$ so $t = \frac{1}{k} \ln \frac{60}{7}$

$t = 24.1$ minutes

Answer: A = 60, k = 0.0892, t = 24.1 minutes

6 Coded data: mean, standard deviation, Σx^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 25 observations x, $\Sigma(x - 20) = 20$ and $\Sigma(x - 20)^2 = 616$.
(a) Find the mean of x.
(b) Find the standard deviation of x.
(c) Find the value of Σx^2.

Mean of $(x - 20) = \frac{20}{25} = 0.8$, so mean of $x = 20.8$

Variance $= \frac{616}{25} - (0.8)^2 = 24$

$\sigma = \sqrt{24}$

$\Sigma x^2 = \Sigma(x - 20)^2 + 40\,\Sigma(x - 20) + 25 \times 20^2$
$= 616 + 800 + 10000$

Answer: (a) 20.8 (b) 4.90 (c) 11416

7 Surds: nested radicals (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\sqrt{39 + 12\sqrt{3}}$ in the form $a + b\sqrt{3}$, where a and b are positive integers.

Suppose $\sqrt{39 + 12\sqrt{3}} = a + b\sqrt{3}$. Square both sides.
$39 + 12\sqrt{3} = a^2 + 3b^2 + 2ab\sqrt{3}$.
Equate: $a^2 + 3b^2 = 39$ and $2ab = 12$, so $ab = 6$.
Try $a = 6$, $b = 1$: $36 + 3 = 39$ ✓.
($a > b\sqrt{3}$, so the root is positive.)

Answer: $6 + 1\sqrt{3}$

8 Particle on a rough table, string slack when B lands (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (3 kg) and B (1 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a rough table with $\mu = 0.25$, and B is 1.6 m above the floor.
The system is released from rest. Find the total distance A moves before coming to rest (A does not reach the pulley).

Until B lands, A: $T - 0.25 \times 3 \times 9.8 = 3a$. B: $1 \times 9.8 - T = 1a$
Adding: $(1 - 0.25 \times 3) \times 9.8 = 4a$, so $a = 0.6125$
$v^2 = 2 \times 0.6125 \times 1.6 = 1.96$, so $v = 1.4$

After B lands, A decelerates at $\mu g = 2.45$: extra distance $= \frac{1.4^2}{2 \times 2.45} = 0.4$

Total distance $= 1.6 + 0.4$

Answer: 2 m

9 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is acute. $\cos A = \frac{21}{29}$ and $\cos B = \frac{15}{17}$.

Find the exact values of $\cos(A + B)$ and $\sin(A - B)$.

$\cos A = \frac{21}{29}$, $\sin A = \frac{20}{29}$; $\sin B = \frac{8}{17}$, $\cos B = \frac{15}{17}$

$\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{155}{493}$

$\sin(A - B) = \sin A \cos B - \cos A \sin B = \frac{132}{493}$

Answer: $\cos(A + B) = \frac{155}{493}$, $\sin(A - B) = \frac{132}{493}$

10 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(9 - x)^{-\frac{1}{2}}$ up to and including the term in x^2.

(b) Use $x = 0.03$ to estimate $\frac{1}{\sqrt{8.97}}$ to 5 decimal places.

(a) $(9 - x)^{-\frac{1}{2}} = \frac{1}{3}(1 - \frac{1}{9}x)^{-\frac{1}{2}}$.

$= \frac{1}{3} + \frac{1}{54}x + \frac{1}{648}x^2 + ...$

(b) $9 - x$ with $x = 0.03$ gives 8.97.

Substitute: $\frac{1}{3} + \frac{1}{54}(0.03) + \frac{1}{648}(0.03)^2$.

$= 0.33389$ (5 d.p.).

Answer: (a) $\frac{1}{3} + \frac{1}{54}x + \frac{1}{648}x^2$ (b) 0.33389