Day 275 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Sinusoidal velocity (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. At time t seconds its velocity is v m s^{-1}, where $v = 3\cos(\frac{\pi t}{2})$ for $t \geq 0$.

Find (a) the greatest magnitude of the acceleration of P, (b) the distance P travels before it first comes to instantaneous rest. Give (b) to 3 s.f.

$a = \frac{dv}{dt} = -\frac{3}{2}\pi \sin(\frac{\pi t}{2})$, so the greatest magnitude is $\frac{3}{2}\pi$

v = 0 when $\cos(\frac{\pi t}{2}) = 0$: first at $\frac{\pi t}{2} = \frac{\pi}{2}$, so t = 1

$s = \int$ from 0 to 1 of $3\cos(\frac{\pi t}{2})\ dt = [\frac{3 \times 2}{\pi}\sin(\frac{\pi t}{2})]$ from 0 to 1

$= \frac{6}{\pi} \times (1 - 0)$

Answer: (a) $\frac{3}{2}\pi$ m s^{-2} (b) 1.91 m

2 Compound change with logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

£5000 is invested at 6% per year compound interest.
Find the least whole number of years for the investment to exceed £15000.

$5000 \times 1.06^n > 15000$, so $1.06^n > 3$

$n > \frac{\ln 3}{\ln 1.06} = 18.85$

n must be a whole number, so n = 19

Answer: 19

3 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ax^n$. The line of best fit for lg y against lg x is lg y = 0.9 + 0.5 lg x.
(a) Find a and n, giving a to 3 significant figures.
(b) Estimate y when x = 100, to 3 significant figures.

$y = ax^n$ gives lg y = lg a + n lg x, so the gradient is n and the intercept is lg a
n = 0.5
lg a = 0.9 so $a = 10^{0.9} = 7.943$
x = 100: $y = 7.943 \times 100^{0.5}$
= 79.4

Answer: (a) a = 7.94, n = 0.5 (b) 79.4

4 Proof by contradiction (irrationality) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\sqrt{13}$ is irrational.

Assume $\sqrt{13} = \frac{a}{b}$ with a, b integers and no common factor.

Then $13b^2 = a^2$, so a^2 is a multiple of 13.
As 13 is prime, a is a multiple of 13: let a = 13k.
Then $13b^2 = 169k^2$, so $b^2 = 13k^2$, and b is a multiple of 13.
a and b have common factor 13: a contradiction. Hence $\sqrt{13}$ is irrational.

Answer: Proved

5 Small angle approximations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (small angle))

θ is small and measured in radians. Use small angle approximations to find the value that

$\frac{\theta \sin 1\theta}{1 - \cos 2\theta}$

approaches as $\theta \to 0$.

$\sin 1\theta \approx 1\theta$ and $1 - \cos 2\theta \approx 2\theta^2$

Expression $\approx \frac{1\theta^2}{2\theta^2} = \frac{1}{2}$

Answer: $\frac{1}{2}$

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6 Poisson: sum of two independent variables, conditional (CIE)

X and Y are independent, X ~ Po(3) and Y ~ Po(2).
(a) State the distribution of X + Y.
(b) Given that X + Y = 4, find the probability that X = 2.

X + Y ~ Po(5)

$P(X = 2 \mid X + Y = 4) = \frac{P(X = 2)P(Y = 2)}{P(X + Y = 4)}$

$= \frac{e^{-3}3^2}{2! \times e^{-2}2^2/2!} \div \frac{e^{-5}5^4}{4!}$

$= C(4, 2)\ (\frac{3}{5})^2\ (\frac{2}{5})^2$, a binomial probability

= 0.346

Answer: (a) Po(5) (b) 0.346

7 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{5x - 6} = x - 4$.

Square both sides: $5x - 6 = (x - 4)^2$.
$x^2 - 13x + 22 = 0$, so x = 2 or x = 11.
Check in $\sqrt{5x - 6} = x - 4$: x = 11 gives $7 \geq 0$ (valid).
x = 2 gives right-hand side $-2 < 0$, so it is rejected.
Answer: x = 11

8 Displacement of the form R sin(ωt + φ) (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a line. Its displacement from a fixed point O at time t seconds is s metres, where s = 12 sin 2t + 5 cos 2t.
Find (a) the greatest distance of P from O, (b) the greatest speed of P, (c) the first time P is at its greatest distance from O (3 s.f.).

Write s = R sin(2t + φ) with R cos φ = 12 and R sin φ = 5

$R = \sqrt{(12^2 + 5^2)} = 13$ and $\tan \varphi = \frac{5}{12}$, so φ = 0.3948 rad

Greatest distance = R = 13

$v = \frac{ds}{dt} = 13 \times 2 \cos(2t + \varphi)$, so greatest speed = 26

s is greatest when $2t + \varphi = \frac{\pi}{2}$: t = (1.5708 − 0.3948) ÷ 2

Answer: (a) 13 m (b) 26 m s^{-1} (c) 0.588 s

9 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is dropped from a height of 2 m. After each bounce it rises to $\frac{3}{5}$ of its previous height.
(a) Find the total distance it travels before it stops.
(b) After how many bounces does it rise less than 0.01 m?

(a) Down H, then up and down each time: $H + 2Hr + 2Hr^2 + ...$

$= H + \frac{2Hr}{1 - r} = 2 + \frac{2 \times 2 \times \frac{3}{5}}{1 - \frac{3}{5}} = 8.$

(b) Rebound height after n bounces $= 2(\frac{3}{5})^n < 0.01$.

$n > \frac{\log(0.005)}{\log(0.6)} = 10.37$, so n = 11.

Answer: (a) 8 m (b) 11 bounces

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10 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^x \sin 2x$. Show that $dy/dx = e^x(\sin 2x + 2\cos 2x)$.
Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (1)e^x \sin 2x + e^x \times 2\cos 2x$
$dy/dx = 0$ requires $1 \sin 2x + 2\cos 2x = 0$
$\tan 2x = -2$, smallest positive $2x = 2.0344$
$x = 1.02$
Second derivative test (or gradient change): a maximum
Answer: x = 1.02, maximum