Day 272 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{6}{x^2 - 6x + 12}$ for all real x.

Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$x^2 - 6x + 12 = (x - 3)^2 + 3$.

$(x - 3)^2 \geq 0$, so the denominator is at least 3, with equality at x = 3.

The denominator is always positive, so f(x) > 0.

Greatest f = $\frac{6}{3} = 2$ at x = 3.

Answer: greatest value 2 at x = 3; range $0 < f(x) \leq 2$

2 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 8 m and mass 30 kg rests on supports at C and D, where AC = 2 m and DB = 3 m. A girl of mass 60 kg stands on it at distance x m from A.

Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $30(1) = 60(x - 5)$, so $x = 5 + \frac{30 \times 1}{60} = 5.5$

Tilting about C (girl before C): $30(2) = 60(2 - x)$, so $x = 2 - \frac{30 \times 2}{60} = 1$

Answer: $1 \leq x \leq 5.5$

3 Logarithmic equations with ln (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $e^x + 15\,e^{-x} = 8$,

giving your answers exactly in terms of natural logarithms.

Multiply by e^x: $e^{2x} - 8e^x + 15 = 0$

$(e^x - 5)(e^x - 3) = 0$

$e^x = 3$ or $e^x = 5$

Answer: $x = \ln 3$ and $x = \ln 5$

4 Equidistant point and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A(1, −5) and B(−3, 3). The point C lies on the x-axis and CA = CB. Find the coordinates of C and the area of triangle ABC.

C = (c, 0): $(c - 1)^2 + 25 = (c - (-3))^2 + 9$.

Expand and solve: $-8c = -8$, so c = 1.

Area = $\frac{1}{2}|(x_A - c)y_B - (x_B - c)y_A|$

$= \frac{1}{2}|0 \times 3 - (-4) \times (-5)| = 10$.

Answer: C(1, 0); area 10

5 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of

$\sin(\sin^{-1}(\frac{5}{13}) + \cos^{-1}(\frac{3}{5}))$.

Let $A = \sin^{-1}(\frac{5}{13})$ and $B = \cos^{-1}(\frac{3}{5})$, both acute

$\sin A = \frac{5}{13}$, $\cos A = \frac{12}{13}$, $\cos B = \frac{3}{5}$, $\sin B = \frac{4}{5}$

$\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{5}{13} \times \frac{3}{5} + \frac{12}{13} \times \frac{4}{5}$

Answer: $\frac{63}{65}$

6 Cubic graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the cubic y = f(x), which touches the x-axis once. Find f(x), giving your answer in expanded form.

Crosses at x = 1 and touches at x = −2, so $f(x) = k(x - 1)(x + 2)^2$.
At (0, −4): $-4 = k(-1)(2)^2 = -4k$, so k = 1.
$f(x) = (x - 1)(x + 2)^2$, then expand.
Answer: $f(x) = x^3 + 3x^2 - 4$

7 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car P passes a point O travelling at a constant speed of 14.7 m s^{-1}. 6 s later, car Q leaves O from rest with constant acceleration 3 m s^{-2}, travelling in the same direction.
Q overtakes P at the point X. Find (a) the time Q takes to reach X, (b) the distance OX, (c) the speed of Q at X.

Let Q travel for t s. P: distance = 14.7(t + 6); Q: distance = ½ × 3 × t^2
Equate: $3t^2 - 29.4t - 176.4 = 0$, so $(t - 14)(3t + 12.6) = 0$
t = 14 (the other root is negative)
OX = 3 × 14^2 / 2 = 294 m, and speed = 3 × 14 = 42
Answer: (a) 14 s (b) 294 m (c) 42 m s^{-1}

8 Poisson: scaling the rate, then binomial (CIE)

Emails arrive in an inbox at an average rate of 4 per hour, independently and at random.
(a) Find the probability of at least 2 in a 20-minute period.
(b) Find the probability that, of 7 separate 20-minute periods, exactly 3 have no arrivals.

20-minute period: $\lambda = 4 \times \frac{20}{60} = \frac{4}{3}$, so $X \sim Po(\frac{4}{3})$
(a) $P(X \geq 2) = 1 - P(X \leq 1) = 0.3849$
(b) $P(X = 0) = e^{-\frac{4}{3}} = 0.2636$
$Y \sim B(7, 0.2636)$: P(Y = 3) = 0.189
Answer: (a) 0.385 (b) 0.189

9 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 + 3xy + 4y^2 = 63$.
Find the coordinates of the points on C where the tangent is parallel to the x-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + 3y}{3x + 8y}$
Horizontal tangent when 2x + 3y = 0, so $y = -\frac{2x}{3}$
Substitute into the equation of C: $x^2 + (3xy) + (4y^2) = 63$
Solve for the remaining coordinate
Answer: (9, −6) and (−9, 6)

10 Critical region, significance level and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A test of H_0: p = 0.25 against H_1: p < 0.25 uses a sample of 30 at the 10% level.
(a) Find the critical region for X, the number of successes.
(b) Find the actual significance level.
(c) Find P(Type II error) when in fact p = 0.05.

$X \sim B(30, 0.25)$. $P(X \leq 4) = 0.0979$ and $P(X \leq 5) = 0.203$
The first is at most 0.1 and the second is not, so the critical region is $X \leq 4$
Actual significance level = $P(X \leq 4) = 0.0979$
Type II error: H_0 is not rejected, so $X \geq 5$, with $X \sim B(30, 0.05)$
$P(X \geq 5) = 1 - P(X \leq 4) = 0.0156$
Answer: (a) $X \leq 4$ (b) 0.0979 (c) 0.0156