Day 270 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $e^{2x} - 8 \times e^{x} + 12 = 0$,
giving each answer exactly and, where not an integer, to 3 s.f.

Let $u = e^{x}$: $u^2 - 8u + 12 = 0$
$(u - 2)(u - 6) = 0$, so $e^{x} = 2$ or $e^{x} = 6$
Take logarithms: $x = \ln 2 = 0.693$; $x = \ln 6 = 1.79$
Answer: $x = \ln 2 = 0.693$; $x = \ln 6 = 1.79$

2 Reading a log graph (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = ax^{n}$. The graph of lg y against lg x is the straight line shown.
Find a and n, giving a to 3 s.f.

$\lg y = \lg a + n \lg x$. The intercept is $\lg a = 1$, so $a = 10^{1} = 10.00$
n = gradient $= \frac{2 - 1}{1} = 1$
Answer: $a = 10.0$, $n = 1$

3 Conditions on the roots of a quadratic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The roots of $x^2 + kx + 12 = 0$ are in the ratio
3 : 1. Find the possible values of k.

Let the roots be α and 3α.
Sum: $4\alpha = -k$. Product: $3\alpha^2 = 12$, so $\alpha^2 = 4$.
$\alpha = \pm 2$, so $k = -4\alpha = \mp 8$.
Answer: $k = \pm 8$

4 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground with speed 63.7 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{5}{12}$. Find (a) the greatest height, (b) the speed when it is at half of this height (3 s.f.). Take g = 9.8 m s^{-2}.

Vertical: $0 = 24.5^2 - 2 \times 9.8 \times H$, so $H = 30.625$
At height $\frac{H}{2} = 15.3125$: $v_y^2 = 24.5^2 - 2 \times 9.8 \times 15.3125 = 300.125$
$v_x = 58.8$ (constant), so speed$^2 = 3457.44 + 300.125 = 3757.565$
Answer: (a) 30.625 m (b) 61.3 m s^{-1}

5 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $n^2 + 6$ is never a multiple of 8, for any integer n.

Consider $n = 8q + r$, with $r = 0, 1, ..., 7$.
Squaring, $n^2 \equiv r^2 \pmod 8$, so:
$0 \to 0$; $1 \to 1$; $2 \to 4$; $3 \to 1$; $4 \to 0$; $5 \to 1$; $6 \to 4$; $7 \to 1$ ($r \to n^2 \bmod 8$)
So $n^2 + 6 \equiv 6$ or 7 or 2 (mod 8).
No residue is 0, so $n^2 + 6$ is never a multiple of 8.
Answer: Proved

Day 270 **A Level Maths: S Tier 10-a-day: Model Solutions**

6 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary points of

$y = \frac{x}{x^2 + 25}$,

and determine the nature of each.

$dy/dx = \frac{(x^2 + 25) - 2x^2}{(x^2 + 25)^2} = \frac{25 - x^2}{(x^2 + 25)^2} = 0$ gives $x = \pm 5$

$y = \pm\frac{1}{10}$ at $x = \pm 5$

d^2y/dx^2 is negative at $x = 5$ and positive at $x = -5$

Answer: $(5, \frac{1}{10})$ maximum; $(-5, -\frac{1}{10})$ minimum

7 Continuous random variable: pdf with unknown constant (CIE)

The pdf of X is $f(x) = kx^2$ for $0 \leq x \leq 5$, and 0 otherwise.
(a) Find k.
(b) Find E(X) and the standard deviation of X.
(c) Find the median, to 3 significant figures.

(a) $\int kx^2\,dx$ from 0 to 5 $= k \times \frac{5^3}{3} = 1$, so $k = \frac{3}{125}$

(b) $E(X) = \int \frac{3}{125} x^3\,dx = \frac{15}{4}$

$E(X^2) = 15$, $Var(X) = 15 - \left(\frac{15}{4}\right)^2 = \frac{15}{16}$, $\sigma = 0.9682$

(c) Median m: $\int f$ from 0 to m $= \frac{m^3}{125} = \frac{1}{2}$

$m = 5 \times (1/2)^{1/3} = 3.969$

Answer: (a) $k = \frac{3}{125}$ (b) $E(X) = \frac{15}{4}$, $\sigma = 0.968$ (c) 3.97

8 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$g(x) = x^2 + 3$. Also $fg(x) = 2x^2 + 7$, where f is a linear function. (a) Find f(x).
(b) Solve $gf(x) = 28$.

(a) Let $f(x) = ax + b$. $fg(x) = a(x^2 + 3) + b = 2x^2 + 7$.
So $a = 2$ and $6 + b = 7$, giving $b = 1$.
$f(x) = 2x + 1$.
(b) $gf(x) = (2x + 1)^2 + 3 = 28$.
$(2x + 1)^2 = 25$, so $2x + 1 = \pm 5$.
$x = -3$ or $x = 2$.
Answer: (a) $f(x) = 2x + 1$ (b) $x = -3$ or $x = 2$

9 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix} -8 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix} -5 \\ 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $-8 - 3s = -5 + 3t$; $1 - 2s = 5$; $s = 3 - 3t$
First two give $s = -2$, $t = 1$, but these fail in the third equation, so the lines do not meet (skew)

$\cos\varphi = \frac{|-12|}{\sqrt{14}\sqrt{18}} = 0.7559$, $\varphi = 40.9°$

Answer: Lines do not meet (skew); angle 40.9°

10 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 6 kg lies on a smooth horizontal table, with a block of mass 6 kg on it ($\mu = 0.5$ between them). A horizontal force P N acts on the plank.

(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 88.2. Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{12}$. The block is accelerated only by friction: $F = 6a \leq \mu mg$

So $a \leq \mu g = 4.9$, and P_max = 12 × 4.9 = 58.8

P = 88.2 > 58.8, so the block slips. Block: $a = \mu g = 4.9$

Plank: 88.2 − 29.4 = 6 a, so a = 9.8

Answer: (a) 58.8 N (b) block 4.9 m s^{-2}, plank 9.8 m s^{-2}