Day 27 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $x^3 = 5x + 2$ has a root α.
(a) Show that the equation can be written as $x = \sqrt[3]{(5x + 2)}$.
(b) Use $x_{n+1} = \sqrt[3]{(5x_n + 2)}$ with $x_0 = 1$ to find x_1, x_2, x_3
to 4 d.p., and hence state α to 3 d.p.

Take the cube root of both sides to obtain $x = \sqrt[3]{(5x + 2)}$
$x_1 = 1.9129$, $x_2 = 2.2614$, $x_3 = 2.3697$
The iteration converges (successive values agree to 3 d.p.): $\alpha = 2.414$
Answer: $x_1 = 1.9129$, $x_2 = 2.2614$, $x_3 = 2.3697$; $\alpha = 2.414$

2 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 8 seats. Each row has 3 more seats than the row in front. Find
(a) the number of seats in row 36,
(b) the least number of rows for at least 1830 seats.

Arithmetic: $a = 8$, $d = 3$.
(a) $u_{36} = 8 + 35 \times 3 = 113$.
(b) $S_n = \frac{n}{2}(16 + 3n - 3) = \frac{n(3n + 13)}{2} \geq 1830$.
$S_{32} = 1744 < 1830$, $S_{33} = 1848 \geq 1830$.
Answer: (a) 113 (b) 33 rows

3 Integrals of the form f'(x)/f(x) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$$\int_0^{\pi/2} \frac{\cos x}{5 + \sin x}\,dx$$

Numerator is the derivative of the denominator
Integral = $[\ln(5 + \sin x)]$
$= \ln 6 - \ln 5$
Answer: $\ln 2 + \ln 3 - \ln 5$

4 Particle in equilibrium on two strings (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P of weight 91 N hangs in equilibrium from two light strings AP and BP attached to a horizontal ceiling at A and B (see diagram), where $\tan\alpha = \frac{3}{4}$ and $\tan\beta = \frac{5}{12}$.
Find (a) the tension in each string, (b) the greatest weight P could have if neither tension may exceed 135 N.

Horizontally: T_1 cos α = T_2 cos β, so $T_1 \times \frac{4}{5} = T_2 \times \frac{12}{13}$
Vertically: $T_1 \sin\alpha + T_2 \sin\beta = 91$, and $T_2 = \frac{13}{15}\,T_1$
$T_1(\frac{3}{5} + \frac{13}{15} \times \frac{5}{13}) = 91$, so $T_1 = 97.5$ and $T_2 = 84.5$
The tension in AP is the larger: it is 97.5 when W = 91
Tensions are proportional to W, so $W_{max} = 91 \times \frac{135}{97.5} = 126$
Answer: (a) AP: 97.5 N, BP: 84.5 N (b) 126 N

5 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 - 4x^2 + 8x - 32$.
Show that $(x - 4)$ is a factor of f(x), and that the equation $f(x) = 0$ has exactly one real root.

$f(4) = 64 - 64 + 32 - 32 = 0$, so $(x - 4)$ is a factor.
$f(x) = (x - 4)(x^2 + 8)$.
For $x^2 + 8 = 0$: $b^2 - 4ac = 0 - 32 = -32 < 0$.
No real roots from the quadratic, so the only real root is $x = 4$.
Answer: Proved; the only real root is $x = 4$

6 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (9 kg) lies on a rough table with $\mu = 0.2$, and is joined by a light string over a smooth pulley (see diagram) to particle B (6 kg) on a smooth slope, $\sin\alpha = \frac{4}{5}$.

Released from rest, the string is cut when B has moved 2 m. Find the further distance A slides before stopping.

A: $T - 0.2 \times 9 \times 9.8 = 9a$. B: $6 \times 9.8 \times \frac{4}{5} - T = 6a$

Adding: $9.8(6 \times \frac{4}{5} - 0.2 \times 9) = 15a$, so $a = 1.96$

$T = 9(1.96 + 0.2 \times 9.8) = 35.28$

When cut: $v^2 = 2 \times 1.96 \times 2 = 7.84$. Then A decelerates at $\mu g = 1.96$

Further distance $= \frac{7.84}{2 \times 1.96} = 2$

Answer: 2 m

7 Waiting for the r-th success (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each shot Amira takes scores with probability $\frac{3}{4}$, independently.

(a) Find the probability that her fourth score comes on shot number 8.
(b) Find the probability that her fourth score comes on or before shot number 8.

(a) fourth score on shot 8: exactly 3 scores in the first 7 shots, then a score

$C(7, 3)(\frac{3}{4})^3(\frac{1}{4})^4 \times \frac{3}{4}$

$= 0.0433$

(b) On or before shot 8 means at least 4 scores in 8 shots: $X \sim B(8, \frac{3}{4})$

$P(X \geq 4) = 1 - P(X \leq 3) = 0.973$

Answer: (a) 0.0433 (b) 0.973

8 Volumes of revolution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The region bounded by $y = xe^{-x}$, the x-axis and the lines $x = 0$ and $x = 1$ is rotated through 2π about the x-axis.
Find the exact volume of the solid formed.

$V = \pi\int_0^1 y^2\,dx = \pi\int_0^1 (xe^{-x})^2\,dx$

Simplify the integrand, then integrate

$V = \pi \times (\frac{1}{4} - \frac{5}{4}e^{-2})$

Answer: $\frac{1}{4}\pi - \frac{5}{4}\pi e^{-2}$

9 Remainder theorem (quadratic divisor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the remainder when x^6 is divided by $(x + 1)(x + 2)$.

$x^6 = (x + 1)(x + 2)Q(x) + px + q$.
$x = -1$: $1 = -p + q$.
$x = -2$: $64 = -2p + q$.
Subtract: $p = -63$, so $p = -63$ and $q = -62$.
Answer: $-63x - 62$

10 Poisson: sum of two independent variables, conditional (CIE)

X and Y are independent, X ~ Po(1) and Y ~ Po(2).

(a) State the distribution of X + Y.

(b) Given that X + Y = 5, find the probability that X = 1.

$X + Y \sim Po(3)$

$$P(X = 1 \mid X + Y = 5) = \frac{P(X = 1)P(Y = 4)}{P(X + Y = 5)}$$

$$= \frac{e^{-1}1^1}{1! \times e^{-2}2^4/4!} \div \frac{e^{-3}3^5}{5!}$$

$= C(5, 1)\ (\frac{1}{3})^1\ (\frac{2}{3})^4$, a binomial probability

$= 0.329$

Answer: (a) Po(3) (b) 0.329