Day 268 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{-6x + 4}{(1 - 3x)(1 - x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x^3, stating the range of validity.

(a) $-6x + 4 \equiv A(1 - x) + B(1 - 3x)$: A = 3, B = 1.

$f(x) = \frac{3}{1 - 3x} + \frac{1}{1 - x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + ...$ (use c = −3, then c = −1).

Coefficients of x^j: $3(3)^j + 1(1)^j$.

Valid for $|x| < \frac{1}{3}$ (both expansions).

Answer: (a) A = 3, B = 1 (b) 4 + 10x + 28x² + 82x³ + ...; $|x| < \frac{1}{3}$

2 Particle in equilibrium on two strings (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P of weight 26 N hangs in equilibrium from two light strings AP and BP attached to a horizontal ceiling at A and B (see diagram), where $\tan\alpha = \frac{12}{5}$ and $\tan\beta = \frac{5}{12}$.

Find (a) the tension in each string, (b) the greatest weight P could have if neither tension may exceed 84 N.

Horizontally: $T_1 \cos\alpha = T_2 \cos\beta$, so $T_1 \times \frac{5}{13} = T_2 \times \frac{12}{13}$

Vertically: $T_1 \sin\alpha + T_2 \sin\beta = 26$, and $T_2 = \frac{5}{12} T_1$

$T_1(\frac{12}{13} + \frac{5}{12} \times \frac{5}{13}) = 26$, so $T_1 = 24$ and $T_2 = 10$

The tension in AP is the larger: it is 24 when W = 26

Tensions are proportional to W, so $W_{max} = 26 \times \frac{84}{24} = 91$

Answer: (a) AP: 24 N, BP: 10 N (b) 91 N

3 Binomial: finding p from a ratio (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(11, p). It is given that P(X = 2) = 1 × P(X = 1).

(a) Find the value of p.

(b) Find P(X = 0), correct to 3 significant figures.

$\frac{110}{2} p^2(1 - p)^9 = 1 \times 11\, p(1 - p)^{10}$

Divide by $p(1 - p)^9$: $5p = 1(1 - p)$

$(5 + 1)p = 1$

$p = \frac{1}{6}$

$P(X = 0) = (1 - p)^{11} = (\frac{5}{6})^{11} = 0.135$

Answer: (a) p = $\frac{1}{6}$ (b) 0.135

4 Sums of multiples (arithmetic series) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the sum of all the integers from 1 to 403 inclusive that are divisible by 5 or by 7, but not by both.

Multiples of 5: 80 terms (5 to 400): sum = 16200.
Multiples of 7: 57 terms: sum = 11571.
Multiples of both, i.e. of 35: 11 terms: sum = 2310.
Required sum = 16200 + 11571 − 2 × 2310.

Answer: 23151

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5 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution u = tan x to find the exact value of

$\int_0^{\pi/3} \sec^2 x \tan^4 x \, dx$

du = $\sec^2 x$ dx; limits become u = 0 and u = tan(π/3)

Integral = $\int_0^{\tan(\pi/3)} u^4 \, du$

Evaluate the limits

Answer: $\frac{9}{5}\sqrt{3}$

6 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + 3n + 3$, $u_1 = 4$.

(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.

(b) Find the least n for which $u_n > 1089$.

$u_n = u_1 + \sum_{k=1}^{n-1}(3k + 3) = 4 + 3 \times \frac{(n-1)n}{2} + 3(n-1).$

$= \frac{3}{2}n^2 + \frac{3}{2}n + 1.$

So $a = \frac{3}{2}$, $b = \frac{3}{2}$, $c = 1$.

$u_{26} = 1054 \leq 1089$ and $u_{27} = 1135 > 1089$.

Answer: (a) $a = \frac{3}{2}$, $b = \frac{3}{2}$, $c = 1$ (b) $n = 27$

7 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $\frac{-5x-9}{(x+3)(x+1)}$ in partial fractions.

(b) Hence find the exact value of $\int_0^2 \frac{-5x-9}{(x+3)(x+1)} \, dx$.

Let N/D = $\frac{A}{x+3} + \frac{B}{x+1}$: $-5x - 9 \equiv A(x+1) + B(x+3)$

x = −3 gives A = −3; x = −1 gives B = −2

Integral = $[-3\ln(x+3) - 2\ln(x+1)]$

Substitute limits 0 and 2

Answer: (a) $-3/(x+3) + (-2)/(x+1)$; (b) $\ln 3 - 3\ln 5$

8 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (1, 4, −1), (0, −2, 4), (2, 0, −2).

Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$\mathrm{BA} = \begin{pmatrix} 1 \\ 6 \\ -5 \end{pmatrix}$, $\mathrm{BC} = \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix}$; BA · BC = 44

$|\mathrm{BA}| = \sqrt{62}$, $|\mathrm{BC}| = \sqrt{44}$

cos ABC = $\frac{44}{\sqrt{62}\sqrt{44}}$ = 0.8424, so ABC = 32.6°

$|\mathrm{BA} \times \mathrm{BC}| = \left|\begin{pmatrix} -26 \\ -4 \\ -10 \end{pmatrix}\right| = \sqrt{792}$

Area = $\frac{1}{2}|\mathrm{BA} \times \mathrm{BC}|$

Answer: ABC = 32.6°, area = $3\sqrt{22}$

9 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (-2t^2 - 2t)\mathbf{i} + (1.5t^2 - 6t)\mathbf{j}$.

Find (a) the value of t at which P is moving parallel to $(2\mathbf{i} + \mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-4t - 2)\mathbf{i} + (3t - 6)\mathbf{j}$

Parallel to $(2\mathbf{i} + \mathbf{j})$: $(-4t - 2) \times 1 = (3t - 6) \times 2$

$t(-10) = -10$, so $t = 1$

At $t = 1$: $\mathbf{v} = (-6\mathbf{i} - 3\mathbf{j})$, speed $= \sqrt{(-6^2 + -3^2)}$

Answer: (a) t = 1 (b) 6.71 m s^{-1}

10 Normal test: critical region and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim N(\mu, 49)$. H_0: $\mu = 82$ is tested against H_1: $\mu > 82$ using the mean of a sample of size 16 at the 10% level.

(a) Find the critical region for the sample mean.

(b) Find the probability of a Type II error when $\mu = 84.5$.

Under H_0, sample mean $\sim N(82, \frac{49}{16})$, standard error = 1.750

$z = 1.2816$ for the upper tail: critical value $= 82 + 1.2816 \times 1.750 = 84.24$

Type II error: do not reject H_0, sample mean < 84.24, where mean $\sim N(84.5, 1.750)$

$z = \frac{84.24 - 84.5}{1.750} = -0.147$

Probability = 0.442

Answer: (a) sample mean > 84.2 (b) 0.442