Day 267 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Boat crossing a river (AQA, Edexcel, OCR, MEI)

A river is 80 m wide and flows at 8 m s^{-1}. A boatman can row at 10 m s^{-1} in still water and wants to reach the point directly opposite his starting point.
Find (a) the angle to the bank at which he should head, to 1 d.p., (b) the time taken, to 3 s.f.

Resultant velocity must be perpendicular to the bank, so the upstream component of the boat's velocity cancels the current
Let the boat head at angle φ upstream from the perpendicular: $10 \sin \varphi = 8$, so $\varphi = 53.1°$
Angle to the bank = $90° - \varphi = 36.9°$
Speed across = $\sqrt{(10^2 - 8^2)} = 6$ m s^{-1}
Time = $\frac{80}{6} = 13.3$ s
Answer: (a) 36.9° (b) 13.3 s

2 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 46 cm and its area is 130 cm^2.
Find the two possible pairs of values of r and θ.

Arc = $r\theta$, so $2r + r\theta = 46$ and $\frac{1}{2}r^2\theta = 130$
$r\theta = 46 - 2r$, so $\frac{1}{2}r(46 - 2r) = 130$
$r^2 - 23r + 130 = 0$, so $(r - 10)(r - 13) = 0$
r = 10: $r\theta = 26$, $\theta = \frac{13}{5}$ r = 13: $r\theta = 20$, $\theta = \frac{20}{13}$
Answer: r = 10, $\theta = \frac{13}{5}$ or r = 13, $\theta = \frac{20}{13}$

3 Poisson approximation to a binomial (CIE)

X ~ B(150, 0.01).
(a) Explain why a Poisson approximation is suitable and state its distribution.
(b) Use it to estimate P(X > 1), to 4 s.f.
(c) Find the exact value of P(X > 1), to 4 s.f.

n = 150 is large and p = 0.01 is small, with np = 1.5
X is approximately Po(1.5)
$P(X > 1) \approx 1 - P(Y \leq 1) = 0.4422$
Exact: $1 - P(X \leq 1)$ with X ~ B(150, 0.01) = 0.4430
Answer: (a) Po(1.5) (b) 0.4422 (c) 0.4430

4 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $e^{2x} - 7 \times e^x + 12 = 0$,
giving each answer exactly and, where not an integer, to 3 s.f.

Let $u = e^x$: $u^2 - 7u + 12 = 0$
$(u - 4)(u - 3) = 0$, so $e^x = 4$ or $e^x = 3$
Take logarithms: $x = \ln 3 = 1.10$; $x = \ln 4 = 1.39$
Answer: x = ln 3 = 1.10; x = ln 4 = 1.39

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5 Combined mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Class Q has 12 students with mean mark 38 and standard deviation 3. Class S has 40 students with mean mark 39 and standard deviation 7.
(a) Find the mean mark of all 52 students.
(b) Find the standard deviation of all 52 students.

Σx for Q = $12 \times 38 = 456$, Σx for S = $40 \times 39 = 1560$

Combined mean = $\frac{2016}{52} = 38.8$

$\Sigma x^2 = n(\sigma^2 + \text{mean}^2)$: $12(9 + 1444) + 40(49 + 1521) = 80236$

Variance = $\frac{80236}{52} - 38.769^2 = 39.947$, so $\sigma = 6.320$

Answer: (a) 38.8 (b) 6.32

6 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of
$\sin(\sin^{-1}(\frac{20}{29}) + \cos^{-1}(\frac{20}{29}))$.

Let A = $\sin^{-1}(\frac{20}{29})$ and B = $\cos^{-1}(\frac{20}{29})$, both acute

$\sin A = \frac{20}{29}$, $\cos A = \frac{21}{29}$, $\cos B = \frac{20}{29}$, $\sin B = \frac{21}{29}$

$\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{20}{29} \times \frac{20}{29} + \frac{21}{29} \times \frac{21}{29}$

Answer: 1

7 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 2x^3 + ax^2 + bx - 16$, where a and b are constants.
$(x - 2)$ is a factor of f(x), and f(x) leaves remainder -60 when divided by $(x + 1)$.
Find a and b, and hence solve $f(x) = 0$.

$f(2) = 0$: $16 + 4a + 2b - 16 = 0$, so $4a + 2b = 0$.
$f(-1) = -60$: $a - b - 18 = -60$.
Solving: $a = -14$, $b = 28$.
$f(x) = 2x^3 - 14x^2 + 28x - 16 = 2(x - 1)(x - 2)(x - 4)$
Answer: a = −14, b = 28; x = 1, 2, 4

8 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{2}{(x + 1)} < \frac{-2}{(x + 2)}$.

Multiply by the positive quantity $(x + 1)^2(x + 2)^2$:
$2(x - (-1))(x - (-2))^2 < (-2)(x - (-2))(x - (-1))^2$.
Move to one side and factorise: $(x - (-1))(x - (-2))(4x + 6) < 0$.
Critical values: -1, -2, $-\frac{3}{2}$. Test the four regions.

Answer: $x < -2$ or $-\frac{3}{2} < x < -1$

9 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 2 kg lies on a smooth horizontal table, with a block of mass 6 kg on it ($\mu = 0.4$ between them). A horizontal force P N acts on the block.

(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 282.24. Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{8}$. The plank is accelerated only by friction: $F = 2a$

Limiting friction: $F = \mu mg = 0.4 \times 6 \times 9.8 = 23.52$, so $a = \frac{23.52}{2} = 11.76$

P_max = $8 \times 11.76 = 94.08$

$P = 282.24 > 94.08$, so the block slips. Plank: $a = \frac{23.52}{2} = 11.76$

Block: $282.24 - 23.52 = 6a$, so $a = 43.12$

Answer: (a) 94.08 N (b) block 43.12 m s^{-2}, plank 11.76 m s^{-2}

10 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{2x + 4}{(1 - x)(1 + 2x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $2x + 4 \equiv A(1 + 2x) + B(1 - x)$: $A = 2$, $B = 2$.

$f(x) = \frac{2}{1 - x} + \frac{2}{1 + 2x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use $c = -1$, then $c = 2$).

Coefficients of x^j: $2(1)^j + 2(-2)^j$.

Valid for $|x| < \frac{1}{2}$ (both expansions).

Answer: (a) A = 2, B = 2 (b) $4 - 2x + 10x^2 - 14x^3 + \ldots$; $|x| < \frac{1}{2}$