Day 266 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Constant speed up a slope, force at an angle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 130 N is pulled at constant speed up a line of greatest slope of a rough plane ($\tan\alpha = \frac{3}{4}$, $\mu = \frac{7}{24}$) by a force P N at angle β above the plane.

(a) Find P when $\tan\beta = \frac{4}{3}$. (b) Find the least possible value of P as β varies.

Perpendicular to the plane: $R + P\sin\beta = 130\cos\alpha$, so $R = 130 \times \frac{4}{5} - P\sin\beta$

Along the plane: $P\cos\beta = 130\sin\alpha + \mu R$, so $P(\cos\beta + \mu\sin\beta) = 130(\sin\alpha + \mu\cos\alpha)$

(a) $P(\frac{3}{5} + \frac{7}{24} \times \frac{4}{5}) = 130(\frac{3}{5} + \frac{7}{24} \times \frac{4}{5})$, so $P = 130$

(b) P is least when $\cos\beta + \mu\sin\beta$ is greatest, i.e. $\sqrt{(1 + \mu^2)} = \frac{25}{24}$ (when $\tan\beta = \frac{7}{24}$)

$P_{min} = 130(\frac{3}{5} + \frac{7}{24} \times \frac{4}{5}) \div \frac{25}{24}$

Answer: (a) 130 N (b) 104 N

2 Tangent to a circle and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle $x^2 + y^2 - 2x - 24 = 0$
passes through P(−2, 4). The tangent at P meets the axes at A and B. Find the area of triangle OAB.

Centre (1, 0). Gradient of radius CP = $-\frac{4}{3}$.

Tangent gradient = $\frac{3}{4}$: $-3x + 4y = 22$.

$x = 0$: $y = \frac{11}{2}$. $y = 0$: $x = -\frac{22}{3}$.

Area = $\frac{1}{2} \times \frac{22}{3} \times \frac{11}{2}$.

Answer: $\frac{121}{6}$

3 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations
$\log_2 x - \log_2 y = 3$
$x + y = 9$

$\log_2(x/y) = 3$, so $x = 2^3y = 8y$
$8y + y = 9$, so $9y = 9$
Answer: x = 8, y = 1

4 PMCC from summary data and a hypothesis test (AQA, Edexcel, OCR, MEI, Eduqas)

For 8 pairs of values, $S_{xx} = 256$, $S_{yy} = 576$ and $S_{xy} = 128$.
(a) Calculate the product moment correlation coefficient r.
(b) The critical value for a one-tailed test at the 5% level with n = 8 is 0.6215. Carry out the test of H_0: $\rho = 0$ against H_1: $\rho > 0$.

$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}} = \frac{128}{\sqrt{(256 \times 576)}} = \frac{128}{384} = 0.3333$
H_0: $\rho = 0$, H_1: $\rho > 0$
$|r| = 0.3333 <$ the critical value 0.6215
Do not reject H_0: there is insufficient evidence of positive correlation between the variables
Answer: (a) r = 0.333 (b) do not reject H_0

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5 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = \frac{5}{2}$ and

$u_{n+1} = \frac{1}{1 - u_n}$.

Show that the sequence is periodic and find

$\sum_{r=1}^{75} u_r$.

$u_1 = \frac{5}{2}$, $u_2 = -\frac{2}{3}$, $u_3 = \frac{3}{5}$, $u_4 = \frac{5}{2}$.

$u_4 = u_1$, so the sequence repeats with period 3.

One period sums to $\frac{73}{30}$.

$75 = 3 \times 25 + 0$: 25 full periods plus the first 0 terms.

Answer: $\mathbf{\frac{365}{6}}$

6 Integration by parts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use integration by parts to find the exact value of

$\int_0^{\pi/2} e^{3x} \sin x \, dx$

Let I be the integral. Apply parts twice (differentiating the trig function each time)

This returns a multiple of I, so $(1 + \frac{1}{9})I$ = boundary terms

$I = \frac{e^{3x}(3\sin x - \cos x)}{10}$

Answer: $\mathbf{\frac{1}{10} + \frac{3}{10}e^{\frac{3\pi}{2}}}$

7 Normal then binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The mass of a apple is $N(76, 9^2)$ g. A apple is 'heavy' if it exceeds 77.

(a) Find the probability that a apple is heavy.

(b) 8 apples are chosen at random. Find the probability that at least 4 are heavy.

(a) $P(X > 77) = P(Z > 0.111) = 0.4558$

(b) Let Y = number heavy, $Y \sim B(8, 0.4558)$

$P(Y \geq 4) = 1 - P(Y \leq 3) = 0.536$

Answer: (a) 0.456 (b) 0.536

8 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:

$x + 2y = 5$

$x^2 + 2xy + y^2 = 36$

From the first equation $x = 5 - 2y$.

Substitute into the second equation and collect terms:

this gives $y^2 - 10y - 11 = 0$.

$y = -1$ or $y = 11$, giving $x = 7$ or $x = -17$.

Answer: (x, y) = (7, −1) and (−17, 11)

9 Ladder against a smooth wall: how high can a person climb (AQA, Edexcel, OCR, MEI)

A uniform ladder AB of length 4 m and weight 150 N has A on rough ground ($\mu = 0.3$) and B against a smooth vertical wall. It makes angle θ with the ground, $\tan\theta = 2.5$.
A man of weight 500 N climbs the ladder. How far along the ladder can he go before it slips?

Wall reaction S horizontal, ground reaction R vertical, friction F = S. Moments about A: $S \times L \sin\theta = W \times \frac{L}{2} \cos\theta$

So $S = \frac{W}{2\tan\theta}$. Unloaded, R = W: need $\frac{W}{2\tan\theta} \le \mu W$, so $\mu \ge \frac{1}{2 \times 2.5} = 0.2$

With the man at distance x: R = 650, and moments about A give $S L \sin\theta = (150 \times 2 + 500x)\cos\theta$

Limiting: $S = \mu R = 0.3 \times 650 = 195$, so $195 \times 4 \times 2.5 = 150 \times 2 + 500x$

$500x = 1950 - 300$, so $x = 3.3$

Answer: 3.3 m

10 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

An open-topped box has a square base of side x m and volume 108 m^3.

Show that the surface area is $A = x^2 + \frac{432}{x}$, and find the minimum value of A, justifying that it is a minimum.

$h = \frac{108}{x^2}$, $A = x^2 + 4xh = x^2 + \frac{432}{x}$

$dA/dx = 2x - \frac{432}{x^2} = 0$, so $x^3 = 216$, $x = 6$

$d^2A/dx^2 = 2 + \frac{864}{x^3} > 0$, so a minimum

$A = 36 + \frac{432}{6}$

Answer: 108 m^2