Day 265 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{-2x+2}{x+5} > 1$.

Rearrange: $\frac{-2x+2}{x+5} - 1 > 0$.

Combine: $\frac{-3x-3}{x+5} > 0$.

Critical values: $x = -1$ (zero of numerator), $x = -5$ (excluded, denominator zero).
Sign analysis of the three regions gives the solution.
Answer: $-5 < x < -1$

2 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population of bacteria is modelled by $P = P_0e^{kt}$, t in hours.
Initially P = 200 and after 3 hours P = 300.
Find k, and the time for the population to reach 2000, to 3 s.f.

$P_0 = 200$. $200\,e^{3k} = 300$ so $e^{3k} = \frac{3}{2}$

$k = \frac{1}{3}\ln\frac{3}{2} = 0.135$

$200\,e^{kt} = 2000$ so $e^{kt} = 10$, $t = \frac{1}{k}\ln 10$

t = 17.0 hours
Answer: k = 0.135, t = 17.0 hours

3 Transforming a graph: images of points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows y = f(x), made of three straight line segments. Find the coordinates of the images of A, B, C and D on the graph of

$y = 2f(-\frac{1}{2}x - 1)$.

Inside f: replace x by $-\frac{1}{2}x - 1$, so x-coordinates become $(x + 1) \div (-\frac{1}{2})$.

Outside: y-coordinates become 2 × y.
A: (2, −8) B: (−2, 10)
C: (−10, −6) D: (−14, −4)
Answer: A′(2, −8), B′(−2, 10), C′(−10, −6), D′(−14, −4)

4 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $e^x = 3x + 2$ has a root α.
(a) Show that the equation can be written as $x = \ln(3x + 2)$.
(b) Use $x_{n+1} = \ln(3x_n + 2)$ with $x_0 = 3$ to find x_1, x_2, x_3
to 4 d.p., and hence state α to 3 d.p.

Take natural logarithms of both sides to obtain $x = \ln(3x + 2)$
$x_1 = 2.3979$, $x_2 = 2.2185$, $x_3 = 2.1582$
The iteration converges (successive values agree to 3 d.p.): $\alpha = 2.125$
Answer: $x_1 = 2.3979$, $x_2 = 2.2185$, $x_3 = 2.1582$; $\alpha = 2.125$

5 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-6-\sqrt{7}}{5+2\sqrt{7}}$ in the form $p + q\sqrt{7}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(5 - 2\sqrt{7})$.
Numerator: $(-6 - \sqrt{7})(5 - 2\sqrt{7}) = -16 + 7\sqrt{7}$.
Denominator: $5^2 - 2^2(7) = -3$.
So the fraction = $\frac{-16+7\sqrt{7}}{-3}$.
Answer: $\frac{16}{3} - \frac{7}{3}\sqrt{7}$

6 Waiting for the r-th success (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each shot Amira takes scores with probability $\frac{1}{2}$, independently.

(a) Find the probability that her third score comes on shot number 10.
(b) Find the probability that her third score comes on or before shot number 10.

(a) third score on shot 10: exactly 2 scores in the first 9 shots, then a score

$C(9, 2)(\frac{1}{2})^2(\frac{1}{2})^7 \times \frac{1}{2}$

$= 0.0352$

(b) On or before shot 10 means at least 3 scores in 10 shots: $X \sim B(10, \frac{1}{2})$

$P(X \geq 3) = 1 - P(X \leq 2) = 0.945$

Answer: (a) 0.0352 (b) 0.945

7 Particle on a rough table, string slack when B lands (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (4 kg) and B (4 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a rough table with $\mu = 0.1$, and B is 0.5 m above the floor.
The system is released from rest. Find the total distance A moves before coming to rest (A does not reach the pulley).

Until B lands, A: $T - 0.1 \times 4 \times 9.8 = 4a$. B: $4 \times 9.8 - T = 4a$
Adding: $(4 - 0.1 \times 4) \times 9.8 = 8a$, so $a = 4.41$
$v^2 = 2 \times 4.41 \times 0.5 = 4.41$, so $v = 2.1$

After B lands, A decelerates at $\mu g = 0.98$: extra distance $= \frac{2.1^2}{2 \times 0.98} = 2.25$

Total distance $= 0.5 + 2.25$

Answer: 2.75 m

8 Cumulative distribution function (CIE)

The cdf of X is $F(x) = 0$ for $x < 0$, $F(x) = 1 - (1 - \frac{x}{5})^4$ for $0 \leq x \leq 5$, and $F(x) = 1$ for $x > 5$.

(a) Find the pdf of X.
(b) Find E(X).
(c) Find $P(1.25 < X < 2.5)$ and the median, to 3 significant figures.

(a) $f(x) = F'(x) = \frac{4}{5}(1 - \frac{x}{5})^3$ for $0 \leq x \leq 5$

(b) $E(X) = \int x\, f(x)\, dx$ from 0 to 5 $= 1$

(c) $P = F(2.5) - F(1.25) = 0.2539$

Median m: $F(m) = 1/2$, so $m = 5(1 - (1/2)^{1/4})$

$m = 0.7955$

Answer: (a) $f(x) = \frac{4}{5}(1 - \frac{x}{5})^3$ (b) 1 (c) 0.254, median 0.796

9 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 5 kg lies on a smooth horizontal table, with a block of mass 5 kg on it ($\mu = 0.2$ between them). A horizontal force P N acts on the block.
(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 29.4. Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{10}$. The plank is accelerated only by friction: $F = 5a$

Limiting friction: $F = \mu mg = 0.2 \times 5 \times 9.8 = 9.8$, so $a = \frac{9.8}{5} = 1.96$

$P_{max} = 10 \times 1.96 = 19.6$

$P = 29.4 > 19.6$, so the block slips. Plank: $a = \frac{9.8}{5} = 1.96$

Block: $29.4 - 9.8 = 5a$, so $a = 3.92$

Answer: (a) 19.6 N (b) block 3.92 m s^{-2}, plank 1.96 m s^{-2}

10 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3 \\ -2 \\ -2 \end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $4 + 3s = -5 - 3t$; $s = -4 - 2t$; $-1 - 2s = 1 - 2t$

Solving the first two: $s = -2$, $t = -1$; the third equation is satisfied

Intersection: $(-2, -2, 3)$

$\cos\varphi = \dfrac{|-7|}{\sqrt{14}\sqrt{17}} = 0.4537$, $\varphi = 63.0°$

Answer: Lines meet at $(-2, -2, 3)$; angle 63.0°