Day 263 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $3x^2 + 3xy + y^2 = 39$.
Find the equation of the tangent to C at the point $(-2, -3)$,
in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x

At $(-2, -3)$: $dy/dx = -\frac{7}{4}$

$y - -3 = (-\frac{7}{4})(x - -2)$, then clear fractions

Answer: 7x + 4y + 26 = 0

2 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 5 and common ratio $\frac{2}{3}$.
Find the least n for which S_n exceeds 95% of the sum to infinity.

$S_n = S_\infty(1 - r^n) > 0.95\, S_\infty$.

$1 - \frac{2}{3}^n > 0.95$, so $\frac{2}{3}^n < 0.05$.

$n > \frac{\log(0.05)}{\log(0.666667)} = 7.388$.

So the least n is 8.

Answer: n = 8

3 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

An arrow is projected from a point O on horizontal ground with speed 24.5 m s^{-1} at an angle α above the horizontal, where $\tan\alpha = \frac{3}{4}$. Find the horizontal distance between the two points at which the arrow is 6.125 m above the ground.
Take g = 9.8 m s^{-2}.

Vertical: $6.125 = 14.7\,t - 4.9t^2$, i.e. $4.9t^2 - 14.7t + 6.125 = 0$

$t = \frac{14.7 \pm \sqrt{(14.7^2 - 19.6 \times 6.125)}}{9.8} = \frac{14.7 \pm 9.8}{9.8}$

The two times differ by 2 s (symmetry about the top of the path)
Horizontal distance = 19.6 × 2

Answer: 39.2 m

4 Critical region, significance level and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A test of H_0: p = 0.5 against H_1: p > 0.5 uses a sample of 50 at the 5% level.
(a) Find the critical region for X, the number of successes.
(b) Find the actual significance level.
(c) Find P(Type II error) when in fact p = 0.6.

X ~ B(50, 0.5). $P(X \geq 32) = 0.0325$ and $P(X \geq 31) = 0.0595$
The first is at most 0.05 and the second is not, so the critical region is $X \geq 32$
Actual significance level = $P(X \geq 32) = 0.0325$
Type II error: H_0 is not rejected, so $X \leq 31$, with X ~ B(50, 0.6)
$P(X \leq 31) = 0.664$

Answer: (a) X ≥ 32 (b) 0.0325 (c) 0.664

5 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population N satisfies $\frac{dN}{dt} = -0.005N^2$, with N = 100 when t = 0.

Find N when t = 18, and the time taken for N to halve.

$\int N^{-2}\, dN = -0.005\int dt$ gives $-\frac{1}{N} = -0.005t + c$

t = 0: $c = -\frac{1}{100}$, so $\frac{1}{N} = 0.005t + \frac{1}{100}$

t = 18: $\frac{1}{N} = \frac{9}{100} + \frac{1}{100}$; halving: $0.005t = \frac{1}{100}$

Answer: N = 10; t = 2

6 Normal then binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The mass of a parcel is $N(80, 8^2)$ kg. A parcel is 'heavy' if it exceeds 86.

(a) Find the probability that a parcel is heavy.

(b) 8 parcels are chosen at random. Find the probability that at least 2 are heavy.

(a) P(X > 86) = P(Z > 0.750) = 0.2266

(b) Let Y = number heavy, Y ~ B(8, 0.2266)

$P(Y \geq 2) = 1 - P(Y \leq 1) = 0.572$

Answer: (a) 0.227 (b) 0.572

7 Circle through three points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle passes through P(−8, −6), Q(6, 8)
and R(0, −10). Find its equation.

Centre is on the perpendicular bisector of each chord.

Bisector of PQ: 14x + 14y = 0.

Bisector of QR: −6x − 18y = 0.

Solving: centre (0, 0).

$r^2 = (-8 - 0)^2 + (-6 - 0)^2 = 100$.

Answer: $(x + 0)^2 + (y + 0)^2 = 100$

8 Transformations: new equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 2x^2 + 4x + 3$ is translated by $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

and then a stretch, scale factor 2, parallel to the x-axis.

Find the equation of the resulting curve in the form $y = ax^2 + bx + c$.

Translation by $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$: $y = f(x - 2) - 1 = 2x^2 - 4x + 2$.

Then a stretch, scale factor 2, parallel to the x-axis: replace x by $\frac{1}{2}x$: $y = f(\frac{1}{2}x - 2) - 1$.

Expand and simplify to give the quadratic.

Answer: $y = \frac{1}{2}x^2 - 2x + 2$

9 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Squares of side x cm are cut from the corners of a square sheet of
side 30 cm and the sides folded up to make an open box.

Find the value of x that maximises the volume, and that maximum volume.

$V = x(30 - 2x)^2 = 4x^3 - 120x^2 + 900x$

$dV/dx = 12x^2 - 240x + 900 = (30 - 2x)(30 - 6x) = 0$

x < 15, so x = 5

$d^2V/dx^2 = 24x - 240$, negative at x = 5: maximum

$V = 5 \times 20^2$

Answer: x = 5, V = 2000 cm^3

10 Hinged rod supported by a string (AQA, Edexcel, OCR, MEI)

A uniform rod AB (3 m, 100 N) is hinged at A to a vertical wall and held horizontal by a string BC, C on the wall above A, with $\tan\alpha = \frac{5}{12}$, where α is the angle between string and rod. A 50 N particle hangs at P, AP = 0.75 m.

Find (a) the tension, (b) the magnitude and direction of the hinge force (3 s.f. and 1 d.p.).

$\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$. Moments about A: $T\sin\alpha \times 3 = 100 \times 1.5 + 50 \times 0.75$

$T \times \frac{5}{13} \times 3 = 187.5$, so $T = 162.5$

Horizontal: $H = T\cos\alpha = 150$. Vertical: $V + T\sin\alpha = 100 + 50$, so $V = 87.5$

Force $= \sqrt{(150^2 + 87.5^2)} = 174$, angle $= \tan^{-1}\left(\frac{87.5}{150}\right) = 30.3°$ above the horizontal

Answer: (a) 162.5 N (b) 174 N at 30.3° above the horizontal (away from the wall)