Day 262 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Horizontal force on a rough slope: range of values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight 90 N rests on a rough plane inclined at α to the horizontal, where $\tan\alpha = \frac{4}{3}$ and $\mu = 0.5$.

A horizontal force H N, in the vertical plane through a line of greatest slope, holds it in equilibrium. Find the range of possible values of H.

$\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$. Normal reaction: $R = 90\cos\alpha + H\sin\alpha$

About to slide down: $H\cos\alpha + \mu R = 90\sin\alpha$, so $H(\frac{3}{5} + 0.5 \times \frac{4}{5}) = 90(\frac{4}{5} - 0.5 \times \frac{3}{5})$

$H_{min} = 45$

About to slide up: $H\cos\alpha = 90\sin\alpha + \mu R$, so $H(\frac{3}{5} - 0.5 \times \frac{4}{5}) = 90(\frac{4}{5} + 0.5 \times \frac{3}{5})$

$H_{max} = 495$

Answer: $45 \leq H \leq 495$

2 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ab^x$. The line of best fit for lg y against x is $\lg y = 0.35 + 0.6x$.

(a) Find a and b to 3 significant figures.

(b) Estimate y when x = 19, to 3 significant figures.

(c) Interpret the value of b.

$y = ab^x$ gives $\lg y = \lg a + x \lg b$

$\lg a = 0.35$ so $a = 10^{0.35} = 2.239$

$\lg b = 0.6$ so $b = 10^{0.6} = 3.981$

$x = 19$: $\lg y = 0.35 + 0.6 \times 19 = 11.75$

$y = 10^{11.75} = 562000000000$

b is the factor y is multiplied by each time x increases by 1 (growth of 300%)

Answer: (a) a = 2.24, b = 3.98 (b) 562000000000 (c) y grows by a factor of 3.98 per unit of x

3 Car, trailer and tow bar (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A car of mass 1500 kg tows a trailer of mass 600 kg along a horizontal road, with driving force 1950 N. Resistances are 300 N on the car and 600 N on the trailer.

(a) Find the tension in the tow bar. At 6 m s^{-1} the tow bar breaks. (b) Find the distance between car and trailer when the trailer stops.

Whole system: $a = \frac{1950 - 300 - 600}{2100} = 0.5$

Trailer: $T - 600 = 600 \times 0.5$, so $T = 900$

After the break the trailer decelerates at $\frac{600}{600} = 1$, and stops after $t = \frac{6}{1} = 6$ s

Trailer travels $\frac{6}{2} \times 6 = 18$ m

Car: $a = \frac{1950 - 300}{1500} = 1.1$, so $s = 6 \times 6 + \frac{1}{2} \times 1.1 \times 6^2 = 55.8$ m

Distance apart = $55.8 - 18$

Answer: (a) 900 N (b) 37.8 m

4 Identity then equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\frac{\sin 2\theta}{1 + \cos 2\theta} \equiv \tan\theta$.

(b) Hence solve $\frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{1}{3}$ for $0 \leq \theta < 360°$ (1 d.p.).

$\frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$\tan\theta = \frac{1}{3}$, so $\theta = 18.4°$ (principal value) and add 180°

Answer: θ = 18.4°, 198.4°

5 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 2 red counters and some black counters, n in total.

Two are taken without replacement. P(at least one red) = $\frac{13}{28}$. Find n.

P(neither red) = $\frac{(n-2)}{n} \times \frac{(n-3)}{(n-1)} = \frac{15}{28}$

$\frac{15}{28}\, n(n-1) = (n-2)(n-3)$

Rearranging gives a quadratic in n whose roots are n = 8 and one other value

The other root is $\frac{21}{13}$ so is rejected (not a valid number of counters)

n = 8

Answer: n = 8

6 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2y + xy^2 = 6$.
Find the equation of the normal to C at the point (1, 2),
in the form ax + by + c = 0 with integers a, b, c.

Differentiate implicitly with respect to x

At (1, 2): dy/dx = $-\frac{8}{5}$

Normal gradient = $-1 \div (-\frac{8}{5}) = \frac{5}{8}$

$y - 2 = \frac{5}{8}(x-1)$, then clear fractions

Answer: 5x − 8y + 11 = 0

7 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx+9}{x-5}$ for $x \neq 5$. Given that f is self-inverse, find k, and hence solve f(x) = x.

Let y = f(x): $y(x-5) = kx + 9$, so $x(y-k) = 5y + 9$.

$f^{-1}(x) = \frac{5x+9}{x-k}$.

$f = f^{-1}$ requires k = 5.

f(x) = x: $5x + 9 = x(x-5)$, so $x^2 - 10x - 9 = 0$.

$x = 5 \pm \sqrt{34} = 5 \pm \sqrt{34}$.

Answer: k = 5; x = $5 \pm \sqrt{34}$

8 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the coefficient of x^4 in the expansion of
$(x-3)(1+4x)^5$.

$(1+4x)^5$: x^4 term has coefficient $C(5,4) \times 4^4 = 1280$, x^3 term has coefficient $C(5,3) \times 4^3 = 640$.

x^4 in the product: $(-3) \times 1280 + 1 \times 640$.

$= -3200$.

Answer: −3200

9 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $\frac{-7x-31}{(x+5)(x+4)}$ in partial fractions.

(b) Hence find the exact value of $\int_0^1 \frac{-7x-31}{(x+5)(x+4)}\, dx$.

Let N/D = $\frac{A}{x+5} + \frac{B}{x+4}$: $-7x - 31 \equiv A(x+4) + B(x+5)$

x = −5 gives A = −4; x = −4 gives B = −3

Integral = $[-4\ln(x+5) - 3\ln(x+4)]$

Substitute limits 0 and 1

Answer: (a) −4/(x + 5) + (−3)/(x + 4); (b) 2ln 2 − 4ln 3 + ln 5

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10 Transformations of y = x^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Describe a sequence of two transformations that maps the curve $y = x^2$ onto the curve $y = 4x^2 - 40x + 107$.

Complete the square: $y = 4(x - 5)^2 + 7$.

Step 1: $y = x^2 \rightarrow y = 4x^2$: a stretch, scale factor 4, parallel to the y-axis.

Step 2: replace x by $(x - 5)$ and add 7: translation by $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$.

Answer: a stretch, scale factor 4, parallel to the y-axis, then translation $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$