Day 261 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Triangular pdf from a diagram (CIE)

The diagram shows the pdf of X.
Find (a) k, (b) E(X), (c) the median of X (3 s.f.).

(a) Area of triangle = ½ × 12 × k = 1, so k = $\frac{1}{6}$

(b) The mean of a triangular distribution is $\frac{0 + 9 + 12}{3} = 7$

F(9) = $\frac{9}{12} = \frac{3}{4} \geq \frac{1}{2}$, so the median is in the left part

F(x) = $\frac{x^2}{108} = \frac{1}{2}$, so x = $\sqrt{54} = 7.348$

Answer: (a) k = $\frac{1}{6}$ (b) E(X) = 7 (c) 7.35

2 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations
$\log_2 x + \log_2 y = 8$
$x - y = 24$

$\log_2(xy) = 8$, so $xy = 2^8 = 256$
$x = y + 24$: $y^2 + 24y - 256 = 0$
$(y - 8)(y + 32) = 0$; $y > 0$ so $y = 8$
Then $x = y + 24$
Answer: x = 32, y = 8

3 Car on a hill: power and maximum speed (CIE)

A car of mass 1200 kg, with engine power 35.52 kW, climbs a straight road with $\sin\alpha = \frac{1}{10}$ against a constant resistance of 600 N.

Find (a) the greatest steady speed up the hill, (b) the acceleration at $\frac{20}{3}$ m s^{-1}. Take g = 9.8 m s^{-2}.

Steady speed: driving force = $600 + 1200 \times 9.8 \times \frac{1}{10} = 1776$ N

P = F v, so v = $\frac{35520}{1776} = 20$

At $\frac{20}{3}$ m s^{-1}: F = $\frac{35520}{\frac{20}{3}} = 5328$ N

F − 1776 = 1200 a, so a = $\frac{3552}{1200}$

Answer: (a) 20 m s^{-1} (b) 2.96 m s^{-2}

4 Modulus inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $|-x + 4| \geq |2x - 3|$.

Square both sides: $(-x + 4)^2 \geq (2x - 3)^2$.
$(-3x + 7)(x + 1) \geq 0$.

Critical values x = $\frac{7}{3}$ and x = −1.

The quadratic has negative coefficient of x^2, so choose the inner region.

Answer: $-1 \leq x \leq \frac{7}{3}$

5 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A cone with its vertex downwards and semi-vertical angle α, where $\tan\alpha = \frac{1}{2}$, is filled at 3π cm^{3}s^{-1}. Find the rate at which the depth h is rising when h = 4 cm. ($V = \frac{1}{3}\pi r^2 h$)

$r = \frac{h}{2}$, so $V = \frac{1}{3}\pi \times \frac{h^2}{4} \times h = \frac{\pi}{12}h^3$

$dV/dt = \frac{\pi}{4}h^2\, dh/dt$, so $3\pi = \frac{\pi}{4} \times 16 \times dh/dt$

$dh/dt = \frac{12}{16}$

Answer: $\frac{3}{4}$ cm s^{-1}

6 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sum_{r=1}^{n}(3r + 6) = 561$, find n.

$\sum_{r=1}^{n}(3r + 6) = 3 \times \frac{n(n + 1)}{2} + 6n.$

$\frac{3\,n(n + 1)}{2} + 6n = 561$, so $3n^2 + 15n - 1122 = 0$.

Solving (n > 0): n = 17.

Answer: n = 17

7 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix}-3\\-6\\-5\end{pmatrix} + s\begin{pmatrix}1\\-2\\-2\end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix}-1\\-7\\0\end{pmatrix} + t\begin{pmatrix}-2\\3\\-3\end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $-3 + s = -1 - 2t$; $-6 - 2s = -7 + 3t$; $-5 - 2s = -3t$

First two give $s = -4$, $t = 3$, but these fail in the third equation, so the lines do not meet (skew)

$\cos\varphi = \frac{|-2|}{\sqrt{9}\sqrt{22}} = 0.1421$, $\varphi = 81.8°$

Answer: Lines do not meet (skew); angle 81.8°

8 Constant speed up a slope, force at an angle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 130 N is pulled at constant speed up a line of greatest slope of a rough plane ($\tan\alpha = \frac{3}{4}$, $\mu = \frac{5}{12}$) by a force P N at angle β above the plane.

(a) Find P when $\tan\beta = \frac{5}{12}$. (b) Find the least possible value of P as β varies.

Perpendicular to the plane: $R + P\sin\beta = 130\cos\alpha$, so $R = 130 \times \frac{4}{5} - P\sin\beta$

Along the plane: $P\cos\beta = 130\sin\alpha + \mu R$, so $P(\cos\beta + \mu\sin\beta) = 130(\sin\alpha + \mu\cos\alpha)$

(a) $P(\frac{12}{13} + \frac{5}{12} \times \frac{5}{13}) = 130(\frac{3}{5} + \frac{5}{12} \times \frac{4}{5})$, so P = 112

(b) P is least when $\cos\beta + \mu\sin\beta$ is greatest, i.e. $\sqrt{(1 + \mu^2)} = \frac{13}{12}$ (when $\tan\beta = \frac{5}{12}$)

$P_{min} = 130(\frac{3}{5} + \frac{5}{12} \times \frac{4}{5}) \div \frac{13}{12}$

Answer: (a) 112 N (b) 112 N

Day 261 **A Level Maths: S Tier 10-a-day: Model Solutions**

9 Conditional probability: independent binomial counts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A biased spinner lands on 'win' with the same unknown probability p each time. It is used 9 times in round 1 and 8 times in round 2.

Given that exactly 6 wins occur altogether, find the probability that exactly 3 of them were in round 1.

Let $X \sim B(9, p)$, $Y \sim B(8, p)$ be independent; $X + Y \sim B(17, p)$

$P(X = 3 \mid X + Y = 6) = \frac{P(X = 3)P(Y = 3)}{P(X + Y = 6)}$

$= \frac{C(9,3)C(8,3)\, p^6(1-p)^{11}}{C(17,6)\, p^6(1-p)^{11}}$

The powers of p and $(1 - p)$ cancel, leaving $\frac{84 \times 56}{12376}$

Answer: $\frac{84}{221} = 0.380$

10 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{-1}{(x-1)} > \frac{-2}{(x-2)}$.

Multiply by the positive quantity $(x - 1)^2(x - 2)^2$:

$(-1)(x - 1)(x - 2)^2 > (-2)(x - 2)(x - 1)^2$.

Move to one side and factorise: $(x - 1)(x - 2)(x) > 0$.

Critical values: 1, 2, 0. Test the four regions.

Answer: $0 < x < 1$ or $x > 2$