Day 26 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Logarithmic equation, reject a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\log_2(x + 2) + \log_2(x - 2) = 5$,
justifying clearly any root that you reject.

$(x + 2)(x - 2) = 2^5 = 32$
$x^2 - 36 = 0$
$(x - 6)(x + 6) = 0$, so $x = 6$ or $x = -6$
$x = -6$ makes $x + 2 = -4 < 0$ and log of a negative number is undefined
Answer: $x = 6$

2 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $n^2(n^2 - 1)$ is divisible by 12
for every positive integer n.

$n^2(n^2 - 1) = n \times (n - 1)n(n + 1)$.
Divisible by 3: one of $n - 1$, n, $n + 1$ is a multiple of 3.
Divisible by 4: if n is even, n^2 is a multiple of 4;
if n is odd, $n - 1$ and $n + 1$ are both even, giving a factor 4.
3 and 4 are coprime, so the expression is divisible by 12.
Answer: Proved

3 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 20 m s^{-1} from the top of a cliff. The top of the cliff is 56.4 m above the sea.
Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take g = 9.8 m s^{-2}.

Take upwards as positive: $-56.4 = 20t - 4.9t^2$, so $4.9t^2 - 20t - 56.4 = 0$

$t = \dfrac{20 + \sqrt{(20^2 + 19.6 \times 56.4)}}{9.8} = \dfrac{20 + 38.8}{9.8} = 6$

Height risen above the cliff top $= \dfrac{20^2}{(2 \times 9.8)} = 20.4$ m

Total distance $= 2 \times 20.4 + 56.4$
Answer: (a) 6 s (b) 97.2 m

4 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(27 + x)^{\frac{1}{3}}$ up to and including the term in x^2.
(b) Use $x = 0.03$ to estimate $\sqrt[3]{27.03}$ to 5 decimal places.

(a) $(27 + x)^{\frac{1}{3}} = 3(1 + \frac{1}{27}x)^{\frac{1}{3}}$.

$= 3 + \frac{1}{27}x - \frac{1}{2187}x^2 + \ldots$

(b) $27 + x$ with $x = 0.03$ gives 27.03.

Substitute: $3 + \frac{1}{27}(0.03) + (-\frac{1}{2187})(0.03)^2$.

= 3.00111 (5 d.p.).

Answer: (a) $3 + \frac{1}{27}x - \frac{1}{2187}x^2$ (b) 3.00111

5 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 22 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 100$, and after 6 minutes $\theta = 74$.
Find A and k, then find when $\theta = 66$ (3 s.f.).

$t = 0$: $22 + A = 100$ so $A = 78$

$t = 6$: $22 + 78e^{-6k} = 74$ so $e^{-6k} = \frac{2}{3}$

$k = \frac{1}{6} \ln \frac{3}{2} = 0.0676$

$78\, e^{-kt} = 44$ so $t = \frac{1}{k} \ln \frac{39}{22}$

$t = 8.47$ minutes

Answer: A = 78, k = 0.0676, t = 8.47 minutes

6 Indices: simultaneous equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously, giving integer x and y:

$4^x \times 16^y = 4096$

$3^x \div 9^y = 9$

Write each side with prime bases: $4096 = 2^{12}$, $9 = 3^2$.
First: $2^{2x} \times 2^{4y} = 2^{12}$, so $2x + 4y = 12$.
Second: $3^{1x - 2y} = 3^2$, so $1x - 2y = 2$.
Solving the pair of linear equations: $x = 4$, $y = 1$.
Answer: x = 4, y = 1

7 Normal: unknown standard deviation then a second probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The lengths of a component, in mm are modelled by $N(93, \sigma^2)$. The probability of a value above 108 is 0.15.
(a) Find σ.
(b) Find the probability of a value below 84.

$P(X > 108) = 0.15$ so $Z = \frac{108 - 93}{\sigma} = 1.0364$

$\sigma = \frac{15}{1.0364} = 14.47$

$z = \frac{84 - 93}{14.47} = -0.622$

$P(X < 84) = P(Z < -0.622) = 0.267$

Answer: (a) 14.5 (b) 0.267

8 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (−3, 3, 0), (−2, −3, −3), (1, 3, 2).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$\mathrm{BA} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}$, $\mathrm{BC} = \begin{pmatrix} 3 \\ 6 \\ 5 \end{pmatrix}$; $\mathrm{BA} \cdot \mathrm{BC} = 48$

$|\mathrm{BA}| = \sqrt{46}$, $|\mathrm{BC}| = \sqrt{70}$

$\cos \mathrm{ABC} = \frac{48}{\sqrt{46}\sqrt{70}} = 0.8459$, so $\mathrm{ABC} = 32.2°$

$|\mathrm{BA} \times \mathrm{BC}| = \left| \begin{pmatrix} 12 \\ 14 \\ -24 \end{pmatrix} \right| = \sqrt{916}$

$\text{Area} = \frac{1}{2}|\mathrm{BA} \times \mathrm{BC}|$

Answer: ABC = 32.2°, area = $\sqrt{229}$

9 Combined mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Class S has 12 students with mean mark 78 and standard deviation 8. Class P has 15 students with mean mark 33 and standard deviation 5.

(a) Find the mean mark of all 27 students.

(b) Find the standard deviation of all 27 students.

Σx for S = $12 \times 78 = 936$, Σx for P = $15 \times 33 = 495$

Combined mean = $\frac{1431}{27} = 53.0$

$\Sigma x^2 = n(\sigma^2 + \text{mean}^2)$: $12(64 + 6084) + 15(25 + 1089) = 90486$

Variance = $\frac{90486}{27} - 53.000^2 = 542.33$, so $\sigma = 23.29$

Answer: (a) 53.0 (b) 23.3

10 Displacement of the form R sin(ωt + φ) (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a line. Its displacement from a fixed point O at time t seconds is s metres, where $s = 6 \sin 2t + 8 \cos 2t$.

Find (a) the greatest distance of P from O, (b) the greatest speed of P, (c) the first time P is at its greatest distance from O (3 s.f.).

Write $s = R \sin(2t + \varphi)$ with $R \cos \varphi = 6$ and $R \sin \varphi = 8$

$R = \sqrt{(6^2 + 8^2)} = 10$ and $\tan \varphi = \frac{8}{6}$, so $\varphi = 0.9273$ rad

Greatest distance = R = 10

$v = \frac{ds}{dt} = 10 \times 2 \cos(2t + \varphi)$, so greatest speed = 20

s is greatest when $2t + \varphi = \frac{\pi}{2}$: $t = (1.5708 - 0.9273) \div 2$

Answer: (a) 10 m (b) 20 m s^{-1} (c) 0.322 s