Day 259 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

An arrow is projected from a point O on horizontal ground with speed 24.5 m s^{-1} at an angle α above the horizontal, where $\tan\alpha = \frac{4}{3}$. Find the horizontal distance between the two points at which the arrow is 14.7 m above the ground. Take g = 9.8 m s^{-2}.

Vertical: $14.7 = 19.6\,t - 4.9t^2$, i.e. $4.9t^2 - 19.6t + 14.7 = 0$

$t = \frac{19.6 \pm \sqrt{(19.6^2 - 19.6 \times 14.7)}}{9.8} = \frac{19.6 \pm 9.8}{9.8}$

The two times differ by 2 s (symmetry about the top of the path)

Horizontal distance = 14.7 × 2

Answer: 29.4 m

2 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both

$x^2 - 3x - 10 \geq 0$ and $2x - 4 \geq 2$.

Quadratic: $(x + 2)(x - 5) \geq 0$ gives $x \leq -2$ or $x \geq 5$.

Linear: $2x - 4 \geq 2$ gives $x \geq 3$.

Both must hold: take the overlap of the two sets.

Answer: $x \geq 5$

3 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A cone with its vertex downwards and semi-vertical angle α, where $\tan\alpha = \frac{1}{2}$, is filled at 15π cm^3s^{-1}. Find the rate at which the depth h is rising when h = 4 cm. ($V = \frac{1}{3}\pi r^2 h$)

$r = \frac{h}{2}$, so $V = \frac{1}{3}\pi \times \frac{h^2}{4} \times h = \frac{\pi}{12}h^3$

$dV/dt = \frac{\pi}{4}h^2\, dh/dt$, so $15\pi = \frac{\pi}{4} \times 16 \times dh/dt$

$dh/dt = \frac{60}{16}$

Answer: $\frac{15}{4}$ cm s^{-1}

4 Particle in equilibrium on two strings (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P of weight 129 N hangs in equilibrium from two light strings AP and BP attached to a horizontal ceiling at A and B (see diagram), where $\tan\alpha = \frac{3}{4}$ and $\tan\beta = \frac{4}{3}$.

Find (a) the tension in each string, (b) the greatest weight P could have if neither tension may exceed 111.2 N.

Horizontally: $T_1 \cos\alpha = T_2 \cos\beta$, so $T_1 \times \frac{4}{5} = T_2 \times \frac{3}{5}$

Vertically: $T_1 \sin\alpha + T_2 \sin\beta = 129$, and $T_2 = \frac{4}{3}\,T_1$

$T_1(\frac{3}{5} + \frac{4}{3} \times \frac{4}{5}) = 129$, so $T_1 = 77.4$ and $T_2 = 103.2$

The tension in BP is the larger: it is 103.2 when W = 129

Tensions are proportional to W, so $W_{max} = 129 \times \frac{111.2}{103.2} = 139$

Answer: (a) AP: 77.4 N, BP: 103.2 N (b) 139 N

5 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{7-2\sqrt{3}}{2+2\sqrt{3}}$ in the form $p + q\sqrt{3}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(2 - 2\sqrt{3})$.
Numerator: $(7 - 2\sqrt{3})(2 - 2\sqrt{3}) = 26 - 18\sqrt{3}$.
Denominator: $2^2 - 2^2(3) = -8$.
So the fraction = $\frac{26 - 18\sqrt{3}}{-8}$.

Answer: $\mathbf{-\frac{13}{4} + \frac{9}{4}\sqrt{3}}$

6 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(30, \frac{1}{4})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{30 - r}{r + 1} \times \frac{1}{3}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(30, r + 1)}{C(30, r)} \times \frac{p}{q} = \frac{30 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{1}{4}$ and $q = \frac{3}{4}$, $p \div q = \frac{1}{3}$

The ratio exceeds 1 when $(30 - r)\frac{1}{4} > (r + 1)\frac{3}{4}$, that is $r < 6.75$

So probabilities rise up to r = 7 and then fall; the mode is 7
P(X = 7) = 0.166
Answer: 7 with probability 0.166

7 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (0, −3, 0), (−2, 4, 5), (1, 5, −1).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix} 2 \\ -7 \\ -5 \end{pmatrix}$, $BC = \begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix}$; $BA \cdot BC = 29$

$|BA| = \sqrt{78}$, $|BC| = \sqrt{46}$

$\cos ABC = \frac{29}{\sqrt{78}\sqrt{46}} = 0.4841$, so $ABC = 61.0°$

$|BA \times BC| = \left|\begin{pmatrix} 47 \\ -3 \\ 23 \end{pmatrix}\right| = \sqrt{2747}$

Area $= \frac{1}{2}|BA \times BC|$

Answer: ABC = 61.0°, area $\mathbf{= \frac{1}{2}\sqrt{2747}}$

8 Partial fractions (improper) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{2x^3 - 6x^2 - 10x + 34}{(x - 3)(x - 2)}$ in the form $ax + b + \frac{C}{(x - 3)} + \frac{D}{(x - 2)}$.

Divide: the numerator has degree 3, the denominator degree 2.
$(2x^3 - 6x^2 - 10x + 34) \div (x^2 - 5x + 6)$ gives quotient $2x + 4$.
Remainder: $-2x + 10$.
Put x = 3: remainder = C(1) so C = 4. Put x = 2: D = −6.

Answer: $\mathbf{2x + 4 + \frac{4}{(x - 3)} - \frac{6}{(x - 2)}}$

9 Triangular pdf from a diagram (CIE)

The diagram shows the pdf of X.

Find (a) k, (b) E(X), (c) the median of X (3 s.f.).

(a) Area of triangle = ½ × 10 × k = 1, so k = $\frac{1}{5}$

(b) The mean of a triangular distribution is $\frac{0 + 6 + 10}{3} = \frac{16}{3}$

F(6) = $\frac{6}{10} = \frac{3}{5} \geq \frac{1}{2}$, so the median is in the left part

F(x) = $\frac{x^2}{60} = \frac{1}{2}$, so x = $\sqrt{30}$ = 5.477

Answer: (a) k = $\frac{1}{5}$ (b) E(X) = $\frac{16}{3}$ (c) 5.48

10 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$.

The point P has coordinates (12, 6, 4).

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathbf{FP} = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix} - t\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$

FP · d = 0: −14 − t(14) = 0, so t = −1

F = (5, 4, 0)

$\mathbf{PF} = \begin{pmatrix} 7 \\ 2 \\ 4 \end{pmatrix}$, so PF = $\sqrt{69}$

Answer: F = (5, 4, 0), PF = $\sqrt{69}$