Day 258 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{10}{2x^2 - 8x + 12}$ for all real x.

Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$2x^2 - 8x + 12 = 2(x - 2)^2 + 4$.

$(x - 2)^2 \geq 0$, so the denominator is at least 4, with equality at x = 2.

The denominator is always positive, so f(x) > 0.

Greatest f = $\frac{10}{4} = \frac{5}{2}$ at x = 2.

Answer: greatest value $\frac{5}{2}$ at x = 2; range $0 < f(x) \leq \frac{5}{2}$

2 Car, trailer and tow bar (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A car of mass 1000 kg tows a trailer of mass 300 kg along a horizontal road, with driving force 1665 N. Resistances are 400 N on the car and 225 N on the trailer.

(a) Find the tension in the tow bar. At 6 m s^{-1} the tow bar breaks. (b) Find the distance between car and trailer when the trailer stops.

Whole system: $a = \frac{1665 - 400 - 225}{1300} = 0.8$

Trailer: $T - 225 = 300 \times 0.8$, so T = 465

After the break the trailer decelerates at $\frac{225}{300} = 0.75$, and stops after $t = \frac{6}{0.75} = 8$ s

Trailer travels $\frac{6}{2} \times 8 = 24$ m

Car: $a = \frac{1665 - 400}{1000} = 1.265$, so $s = 6 \times 8 + \frac{1}{2} \times 1.265 \times 8^2 = 88.48$ m

Distance apart = 88.48 − 24

Answer: (a) 465 N (b) 64.48 m

3 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{4x^2 - 20x + 29}{(x - 3)(x - 2)^2}$ in the form

$\frac{A}{(x - 3)} + \frac{B}{(x - 2)} + \frac{C}{(x - 2)^2}$.

$4x^2 - 20x + 29 \equiv A(x - 2)^2 + B(x - 3)(x - 2) + C(x - 3)$.

Put x = 3: $5 = A(1)^2$, so A = 5.

Put x = 2: $5 = C(-1)$, so C = −5.

Compare x^2: $4 = 1A + 1B$, so B = −1.

Answer: A = 5, B = −1, C = −5

4 Stratified sample then probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A school has 180 students in Year 10, 60 in Year 11 and 240 in the sixth form.

A stratified sample of 40 students is taken.

(a) Find the number sampled from each group.

(b) Two students are chosen at random from the sample. Find the probability they are from the same group.

Sample sizes: $40 \times \frac{180}{480}$, $40 \times \frac{60}{480}$, $40 \times \frac{240}{480} = 15, 5, 20$

P(same) = 15 × 14 + 5 × 4 + 20 × 19/40 × 39

$= \frac{610}{1560} = 0.391$

Answer: (a) 15, 5, 20 (b) 0.391

5 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is acute. $\sin A = \frac{5}{13}$ and $\sin B = \frac{4}{5}$.

Find the exact values of $\tan(A - B)$ and $\cos(A - B)$.

$\cos A = \frac{12}{13}$, $\sin A = \frac{5}{13}$; $\sin B = \frac{4}{5}$, $\cos B = \frac{3}{5}$

$\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{-\frac{33}{65}}{\frac{56}{65}} = -\frac{33}{56}$

$\cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{56}{65}$

Answer: $\tan(A - B) = -\frac{33}{56}$, $\cos(A - B) = \frac{56}{65}$

6 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sum_{r=1}^{n}(2r - 2) = 132$, find n.

$\sum_{r=1}^{n}(2r - 2) = 2 \times \frac{n(n + 1)}{2} - 2n.$

$\frac{2\,n(n + 1)}{2} - 2n = 132$, so $2n^2 - 2n - 264 = 0$.

Solving $(n > 0)$: $n = 12$.

Answer: n = 12

7 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = x^2 + 4$ to find the exact value of

$\int_0^1 x(x^2 + 4)^4\,dx$

$du = 2x\,dx$; limits $x = 0, 1$ become $u = 4, 5$

Integral $= \frac{1}{2}\int_4^5 u^4\,du = [\frac{u^5}{10}]$

Evaluate the limits

Answer: $\frac{2101}{10}$

8 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two drones P and Q move so that, t seconds after launch, their position vectors in metres are

$\mathbf{r}_P = \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix} 4 \\ -11 \\ 1 \end{pmatrix} + t\begin{pmatrix} -6 \\ 1 \\ 6 \end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} 0 \\ -7 \\ -4 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} = 0$

$-19 + t(19) = 0$, so $t = 1$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} -3 \\ -6 \\ -1 \end{pmatrix}$, with length $\sqrt{46}$

Answer: t = 1, minimum distance = $\sqrt{46}$ m

9 Critical region, significance level and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A test of H_0: p = 0.1 against H_1: p > 0.1 uses a sample of 20 at the 1% level.

(a) Find the critical region for X, the number of successes.

(b) Find the actual significance level.

(c) Find P(Type II error) when in fact p = 0.2.

X ~ B(20, 0.1). P(X ≥ 7) = 0.00239 and P(X ≥ 6) = 0.0113

The first is at most 0.01 and the second is not, so the critical region is X ≥ 7

Actual significance level = P(X ≥ 7) = 0.00239

Type II error: H_0 is not rejected, so X ≤ 6, with X ~ B(20, 0.2)

P(X ≤ 6) = 0.913

Answer: (a) X ≥ 7 (b) 0.00239 (c) 0.913

10 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector (12**i** + 12**j**) km relative to a port O, and moves with constant velocity (2**i** + 4**j**) km h^{-1}. Ship B leaves O at t = 0 with constant speed 10 km h^{-1} and intercepts A.

Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: (12**i** + 12**j**) + t(2**i** + 4**j**). B at time t: t v_B (unknown velocity)

At interception B has travelled 10 t, so |(12**i** + 12**j**) + t(2**i** + 4**j**)| = 10 t

Squaring and rearranging: $-80t^2 + 144t + 288 = 0$, which gives t = 3 or a negative value

t = 3 (the other root is negative since B is faster than A)

A is then at (12**i** + 12**j**) + 3(2**i** + 4**j**) = (18**i** + 24**j**), so v_B = $\frac{1}{3}$(18**i** + 24**j**) = (6**i** + 8**j**)

Answer: (a) 3 h (b) (6i + 8j) km h^{-1}