Day 257 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Velocity with a square-root term (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. At time t seconds, where $t \geq 0$, its velocity is v m s^{-1}, where $v = 16\sqrt{t} - 4t$.
Find (a) the greatest speed of P, (b) the greatest displacement of P from its starting point.

$\frac{dv}{dt} = \frac{16}{2\sqrt{t}} - 4 = 0$ when $\sqrt{t} = 2$, so $t = 4$
$v = 16 \times 2 - 4 \times 4 = 16$
$v = 0$ when $\sqrt{t}(16 - 4\sqrt{t}) = 0$: $t = 16$ ($t = 0$ is the start)
$s = \int v\,dt = \frac{2}{3} \times 16\, t^{\frac{3}{2}} - \frac{4}{2}t^2$, so at $t = 16$: $s = \frac{2048}{3} - 512$

Answer: (a) 16 m s^{-1} (b) $\frac{512}{3}$ m

2 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 5 strips to estimate
$\int_1^2 x\, e^{-x}\, dx$, to 4 s.f.
Is it an overestimate or an underestimate? Explain.

$h = \frac{1}{5}$; ordinates at $x = 1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \ldots, 2$

Sum $= \frac{h}{2}[y_0 + y_5 + 2(y_1 + \ldots + y_4)] = 0.3293$

$f''(x) < 0$ on the interval, so the curve is concave and the trapezia lie below the curve
Answer: 0.3293; underestimate

3 Circle touching the axes (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle in the diagram touches both axes and passes through P. Find the equation of the circle, giving it in the form $x^2 + y^2 + ax + by + c = 0$.

Touches both axes in the first quadrant: centre (r, r), radius r.
P(4, 2) lies on the circle: $(4 - r)^2 + (2 - r)^2 = r^2$.
$r^2 - 12r + 20 = 0$, so $r = 2$ or $r = 10$.
The diagram shows P on the smaller circle, so $r = 2$.
$(x - 2)^2 + (y - 2)^2 = 4$, then expand.
Answer: $x^2 + y^2 - 4x - 4y + 4 = 0$

4 Logarithmic equations with ln (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $e^x + 6\, e^{-x} = 5$,
giving your answers exactly in terms of natural logarithms.

Multiply by e^x: $e^{2x} - 5e^x + 6 = 0$
$(e^x - 3)(e^x - 2) = 0$
$e^x = 2$ or $e^x = 3$
Answer: $x = \ln 2$ and $x = \ln 3$

5 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car A passes a point O travelling at a constant speed of 13.5 m s^{-1}. 10 s later, car B leaves O from rest with constant acceleration 3 m s^{-2}, travelling in the same direction.
B overtakes A at the point X. Find (a) the time B takes to reach X, (b) the distance OX, (c) the speed of B at X.

Let B travel for t s. A: distance $= 13.5(t + 10)$; B: distance $= \frac{1}{2} \times 3 \times t^2$
Equate: $3t^2 - 27t - 270 = 0$, so $(t - 15)(3t + 18) = 0$
$t = 15$ (the other root is negative)
OX $= 3 \times 15^2 / 2 = 337.5$ m, and speed $= 3 \times 15 = 45$
Answer: (a) 15 s (b) 337.5 m (c) 45 m s^{-1}

6 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 25 and upper quartile 31. An outlier is a value more than 1.5 × IQR beyond the nearest quartile.

(a) Find the range of values that are not outliers.

(b) 8 observations have mean 28. The largest is 47, an outlier, and it is removed. Find the new mean.

IQR = 31 − 25 = 6, so 1.5 × IQR = 9

Lower limit = 25 − 9 = 16, upper limit = 31 + 9 = 40

$\Sigma x = 8 \times 28 = 224$, so without 47: $\Sigma x = 177$

New mean = $\frac{177}{7}$

Answer: (a) 16 to 40 (b) 25.29

7 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(16 - x)^{-\frac{1}{2}}$ up to and including the term in x^2.

(b) Use $x = 0.02$ to estimate $\frac{1}{\sqrt{15.98}}$ to 5 decimal places.

(a) $(16 - x)^{-\frac{1}{2}} = \frac{1}{4}(1 - \frac{1}{16}x)^{-\frac{1}{2}}$.

$= \frac{1}{4} + \frac{1}{128}x + \frac{3}{8192}x^2 + ...$

(b) $16 - x$ with $x = 0.02$ gives 15.98.

Substitute: $\frac{1}{4} + \frac{1}{128}(0.02) + \frac{3}{8192}(0.02)^2$.

= 0.25016 (5 d.p.).

Answer: (a) $\frac{1}{4} + \frac{1}{128}x + \frac{3}{8192}x^2$ (b) 0.25016

8 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (5, 4, 3), (−1, 4, −2), (−2, 3, −1).

Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix} 6 \\ 0 \\ 5 \end{pmatrix}$, $BC = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$; $BA \cdot BC = -1$

$|BA| = \sqrt{61}$, $|BC| = \sqrt{3}$

$\cos ABC = \frac{-1}{\sqrt{61}\sqrt{3}} = -0.0739$, so ABC = 94.2°

$|BA \times BC| = \left|\begin{pmatrix} 5 \\ -11 \\ -6 \end{pmatrix}\right| = \sqrt{182}$

Area = $\frac{1}{2}|BA \times BC|$

Answer: ABC = 94.2°, area = $\frac{1}{2}\sqrt{182}$

9 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{11x + 5}{(1 - x)(1 + 3x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $11x + 5 \equiv A(1 + 3x) + B(1 - x)$: A = 4, B = 1.

$f(x) = \frac{4}{1 - x} + \frac{1}{1 + 3x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + ...$ (use c = −1, then c = 3).

Coefficients of x^j: $4(1)^j + 1(-3)^j$.

Valid for $|x| < \frac{1}{3}$ (both expansions).

Answer: (a) A = 4, B = 1 (b) $5 + x + 13x^2 - 23x^3 + ...$; $|x| < \frac{1}{3}$

10 Normal: unknown standard deviation then a second probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The lengths of a component, in mm are modelled by $N(99, \sigma^2)$. The probability of a value below 90 is 0.20.

(a) Find σ.

(b) Find the probability of a value below 124.

$P(X < 90) = 0.20$ so $Z = \frac{90 - 99}{\sigma} = -0.8416$

$\sigma = \frac{-9}{-0.8416} = 10.69$

$z = \frac{124 - 99}{10.69} = 2.338$

$P(X < 124) = P(Z < 2.338) = 0.990$

Answer: (a) 10.7 (b) 0.990