Day 255 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Partial fractions (improper) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{x^3 - 4x^2 + 9x - 12}{(x-2)(x-3)}$ in the form $ax + b + \frac{C}{(x-2)} + \frac{D}{(x-3)}$.

Divide: the numerator has degree 3, the denominator degree 2.

$(x^3 - 4x^2 + 9x - 12) \div (x^2 - 5x + 6)$ gives quotient $x + 1$.

Remainder: $8x - 18$.

Put $x = 2$: remainder $= C(-1)$ so $C = 2$. Put $x = 3$: $D = 6$.

Answer: $x + 1 + \frac{2}{(x-2)} + \frac{6}{(x-3)}$

2 Logarithmic equations with ln (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2 \ln x = \ln(x + 6) + \ln 3$,
stating clearly why one root is rejected.

$\ln x^2 = \ln(3(x + 6))$, so $x^2 = 3x + 18$

$x^2 - 3x - 18 = 0$, $(x - 6)(x + 3) = 0$

$x = -3$ is rejected since $\ln x$ is undefined for $x < 0$

Answer: $x = 6$

3 Arithmetic terms forming a geometric sequence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first term of an arithmetic sequence is 2 and its common difference d is non-zero. The 1st, 2nd and 12th terms are the first three terms of a geometric sequence. Find d and the common ratio.

Terms: 2, $2 + d$, $2 + 11d$.

Geometric: $(2 + d)^2 = (2)(2 + 11d)$.

Expand: $d^2 + 4d + 4 = 22d + 4$, so $d(d - 18) = 0$.

$d \neq 0$, so $d = 18$. Ratio $= \frac{20}{2} = 10$.

Answer: $d = 18$, ratio $= 10$

4 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = e^x + 3x - 14$.

(a) Show that $f(x) = 0$ has a root α with $2 < \alpha < 3$.

(b) Use Newton-Raphson with $x_0 = 2$ to find x_1 and x_2
to 4 decimal places.

$f(2) = -0.611$ and $f(3) = 15.1$: a change of sign, f continuous

$f'(x) = e^x + 3$, so $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = 2 - \frac{-0.6109}{10.39} = 2.0588$

$x_2 = 2.0588 - \frac{0.01303}{10.84} = 2.0576$

Answer: $x_1 = 2.0588$, $x_2 = 2.0576$

5 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each component is faulty with probability 0.03, independently.
Find the least number n of components that must be tested so that the probability of finding at least one faulty is greater than 0.95.

P(at least one faulty) $= 1 - 0.97^n > 0.95$

$0.97^n < 0.05$

$n \ln 0.97 < \ln 0.05$

Dividing by $\ln 0.97$ (negative) reverses the inequality: $n > 98.35$

$n = 99$

Answer: $n = 99$

6 Derivative of the inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve is given by $x = \ln y + y^2$.

Find dy/dx and d^2y/dx^2 when $y = 1$.

$dx/dy = 2y + \frac{1}{y} = 3$ when $y = 1$, so $dy/dx = \frac{1}{dx/dy} = \frac{1}{3}$

$d^2x/dy^2 = 1$ when $y = 1$

$d^2y/dx^2 = -\frac{d^2x}{dy^2/(dx/dy)^3} = -\frac{1}{3^3}$

Answer: $dy/dx = \frac{1}{3}$, $d^2y/dx^2 = -\frac{1}{27}$

7 Hinged rod supported by a string (AQA, Edexcel, OCR, MEI)

A uniform rod AB (3 m, 40 N) is hinged at A to a vertical wall and held horizontal by a string BC, C on the wall above A, with $\tan\alpha = \frac{5}{12}$, where α is the angle between string and rod. A 50 N particle hangs at P, AP = 2.25 m.

Find (a) the tension, (b) the magnitude and direction of the hinge force (3 s.f. and 1 d.p.).

$\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$. Moments about A: $T\sin\alpha \times 3 = 40 \times 1.5 + 50 \times 2.25$

$T \times \frac{5}{13} \times 3 = 172.5$, so $T = 149.5$

Horizontal: $H = T\cos\alpha = 138$. Vertical: $V + T\sin\alpha = 40 + 50$, so $V = 32.5$

Force $= \sqrt{(138^2 + 32.5^2)} = 142$, angle $= \tan^{-1}(\frac{32.5}{138}) = 13.3°$ above the horizontal

Answer: (a) 149.5 N (b) 142 N at 13.3° above the horizontal (away from the wall)

8 Tree diagram with an unknown probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tree diagram shows two games played in turn.

Given the second is a H, P(first was a H) $= \frac{1}{7}$. Find x.

P(H then H) $= 0.2x$, P(second H) $= 0.2x + 0.4(1 - x)$

P(first H | second H) $= \frac{0.2x}{0.2x + 0.4(1 - x)} = \frac{1}{7}$

$0.2x = \frac{1}{7}(0.2x + 0.4 - 0.4x)$

Solve the linear equation for x: $x = \frac{1}{4}$

Answer: $x = \frac{1}{4}$

9 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{2x + 9}{(1 + 2x)(1 - 2x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $2x + 9 \equiv A(1 - 2x) + B(1 + 2x)$: $A = 4$, $B = 5$.

$f(x) = \frac{4}{1 + 2x} + \frac{5}{1 - 2x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use $c = 2$, then $c = -2$).

Coefficients of x^j: $4(-2)^j + 5(2)^j$.

Valid for $|x| < \frac{1}{2}$ (both expansions).

Answer: (a) $A = 4$, $B = 5$ (b) $9 + 2x + 36x^2 + 8x^3 + \ldots$; $|x| < \frac{1}{2}$

10 Non-uniform rod: centre of mass and tilting (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB of length 6 m and mass 8 kg rests on supports at C and D, where AC = 1 m and AD = 5 m, with $R_C = 1.5\,R_D$.

Find (a) the distance of the centre of mass from A. A particle of mass m kg at B puts the rod on the point of tilting about D. (b) Find m.

$R_C = 1.5\,R_D$. Moments about A: $R_C \times 1 + R_D \times 5 = 8\,g \times AG$

$R_D(1.5 \times 1 + 5) = 8\,g \times AG$ with $R_D(1 + 1.5) = 8\,g$, so $AG = \dfrac{1.5 \times 1 + 5}{2.5} = 2.6$

Tilting about D: $R_C = 0$. Moments about D: $m\,g\,(6 - 5) = 8\,g\,(5 - 2.6)$

$m = \dfrac{8 \times 2.4}{1} = 19.2$

Answer: (a) 2.6 m (b) 19.2