Day 253 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 36 cm and its area is 65 cm^2.
Find the two possible pairs of values of r and θ.

Arc = $r\theta$, so $2r + r\theta = 36$ and $\frac{1}{2}r^2\theta = 65$

$r\theta = 36 - 2r$, so $\frac{1}{2}r(36 - 2r) = 65$

$r^2 - 18r + 65 = 0$, so $(r - 5)(r - 13) = 0$

$r = 5$: $r\theta = 26$, $\theta = \frac{26}{5}$ $r = 13$: $r\theta = 10$, $\theta = \frac{10}{13}$

Answer: $r = 5$, $\theta = \frac{26}{5}$ or $r = 13$, $\theta = \frac{10}{13}$

2 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level with speed 78.4 m s^{-1}.
Find the length of time for which the ball is more than 137.2 m above the ground, and the greatest height it reaches. Take $g = 9.8$ m s^{-2}.

At height 137.2 m: $v^2 = 78.4^2 - 2 \times 9.8 \times 137.2 = 3457.44$, so $v = 58.8$

Time from this height to the top: $t = \frac{58.8}{9.8} = 6$ s

By symmetry the time above this height is $2 \times 6 = 12$ s

Greatest height = $\frac{78.4^2}{(2 \times 9.8)} = 313.6$

Answer: 12 s, 313.6 m

3 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(25 + x)^{\frac{1}{2}}$ up to and including the term in x^2.
(b) Use $x = 0.04$ to estimate $\sqrt{25.04}$ to 5 decimal places.

(a) $(25 + x)^{\frac{1}{2}} = 5(1 + \frac{1}{25}x)^{\frac{1}{2}}$.

$= 5 + \frac{1}{10}x - \frac{1}{1000}x^2 + ...$

(b) $25 + x$ with $x = 0.04$ gives 25.04.

Substitute: $5 + \frac{1}{10}(0.04) + (-\frac{1}{1000})(0.04)^2$.

= 5.00400 (5 d.p.).

Answer: (a) $5 + \frac{1}{10}x - \frac{1}{1000}x^2$ (b) 5.00400

4 Segments: exact area and perimeter (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A chord AB of a circle, centre O and radius 3 cm, subtends an angle of $\frac{2\pi}{3}$ at O.
Find the exact area and the exact perimeter of the minor segment cut off by AB.

Sector OAB: $\frac{1}{2}r^2\theta = \frac{1}{2} \times 3^2 \times \frac{2\pi}{3}$; triangle OAB: $\frac{1}{2}r^2\sin\theta = \frac{9}{4}\sqrt{3}$

Area = $3\pi - \frac{9}{4}\sqrt{3}$

Chord AB = $2r\sin(\theta/2) = 3\sqrt{3}$
Arc AB = $r\theta = 2\pi$
Perimeter = arc + chord

Answer: Area = $3\pi - \frac{9}{4}\sqrt{3}$ cm^2, perimeter = $2\pi + 3\sqrt{3}$ cm

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5 Conditions on the roots of a quadratic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The roots of $x^2 + kx + 32 = 0$ are in the ratio 2 : 1. Find the possible values of k.

Let the roots be α and 2α.

Sum: $3\alpha = -k$. Product: $2\alpha^2 = 32$, so $\alpha^2 = 16$.

$\alpha = \pm 4$, so $k = -3\alpha = \mp 12$.

Answer: $k = \pm 12$

6 Compound change with logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drug dose of 20 mg decays by 10% each hour. Find the first whole number of hours after which less than 2 mg remains.

$20 \times 0.90^n < 2$, so $0.90^n < \frac{1}{10}$

$n > \frac{\ln \frac{1}{10}}{\ln 0.90} = 21.85$ ($\ln 0.90$ is negative, so the inequality reverses)

$n = 22$

Answer: 22

7 Poisson hypothesis test (CIE)

Faults in a cable occur at random at a mean rate of 3 per hour. After a modification, 0 faults are found in 1 hour. Test at the 10% level whether the rate of faults has decreased.

H_0: $\lambda = 3$ per hour, H_1: $\lambda < 3$ per hour

Over 1 hour: $X \sim Po(3)$

$P(X \leq 0) = 0.0498$

$0.0498 < 10\%$, so reject H_0

There is sufficient evidence that the rate has decreased

Answer: Reject H_0 (p-value 0.0498)

8 Critical region, significance level and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A test of H_0: $p = 0.25$ against H_1: $p > 0.25$ uses a sample of 25 at the 5% level.

(a) Find the critical region for X, the number of successes.

(b) Find the actual significance level.

(c) Find P(Type II error) when in fact $p = 0.45$.

$X \sim B(25, 0.25)$. $P(X \geq 11) = 0.0297$ and $P(X \geq 10) = 0.0713$

The first is at most 0.05 and the second is not, so the critical region is $X \geq 11$

Actual significance level = $P(X \geq 11) = 0.0297$

Type II error: H_0 is not rejected, so $X \leq 10$, with $X \sim B(25, 0.45)$

$P(X \leq 10) = 0.384$

Answer: (a) $X \geq 11$ (b) 0.0297 (c) 0.384

9 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = 4x^2 - 8x - 1$ for $x \leq k$. Given that k is the greatest value for which f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = 4(x - 1)^2 - 5$.

The vertex is at $x = 1$, so $k = 1$ (f is one-one on one side of it).

Let $y = f(x)$: $y + 5 = 4(x - 1)^2$, so $(x - 1) = \pm\sqrt{\frac{y + 5}{4}}$.

As $x \leq 1$, take the negative root: $x = 1 - \sqrt{\frac{y + 5}{4}}$.

Range of f = domain of f^{-1}: $f(1) = -5$, so $x \geq -5$.

Answer: $k = 1$; $f^{-1}(x) = 1 - \sqrt{\frac{x + 5}{4}}$; $x \geq -5$

10 Boat crossing a river (AQA, Edexcel, OCR, MEI)

A river is 120 m wide and flows at 34 m s^{-1}. A boat has speed 16 m s^{-1} in still water and crosses the river heading so that it is carried as little distance downstream as possible.
Find the least possible distance downstream between the starting point and the landing point.

The boat speed is less than the current, so it cannot stop drifting. Let the boat head at angle φ upstream from the perpendicular to the bank

Across: 16 cos φ; along (downstream): 34 − 16 sin φ. Drift = $\frac{120}{16\cos\varphi}(34 - 16\sin\varphi)$

Drift is least when the resultant velocity is at the greatest angle to the downstream direction: the boat velocity is perpendicular to the resultant, so $\sin\varphi = \frac{16}{34}$

Then across speed = $\frac{240}{17}$ and downstream speed = $\frac{450}{17}$, so drift = $120 \times \frac{\frac{450}{17}}{\frac{240}{17}}$

Answer: 225 m