Day 252 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $e^{2x} - 15 \times e^x + 56 = 0$,
giving each answer exactly and, where not an integer, to 3 s.f.

Let $u = e^x$: $u^2 - 15u + 56 = 0$

$(u - 7)(u - 8) = 0$, so $e^x = 7$ or $e^x = 8$

Take logarithms: $x = \ln 7 = 1.95$; $x = \ln 8 = 2.08$

Answer: $x = \ln 7 = 1.95$; $x = \ln 8 = 2.08$

2 Binomial: finding p from a ratio (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(6, p)$. It is given that $P(X = 2) = \frac{1}{2} \times P(X = 1)$.

(a) Find the value of p.
(b) Find $P(X = 0)$, correct to 3 significant figures.

$\frac{30}{2} p^2(1 - p)^4 = \frac{1}{2} \times 6\, p(1 - p)^5$

Divide by $p(1 - p)^4$: $2.5\, p = \frac{1}{2}(1 - p)$

$(2.5 + \frac{1}{2})p = \frac{1}{2}$

$p = \frac{1}{6}$

$P(X = 0) = (1 - p)^6 = (\frac{5}{6})^6 = 0.335$

Answer: (a) $p = \frac{1}{6}$ (b) 0.335

3 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 18 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 90$, and after 8 minutes $\theta = 54$.
Find A and k, then find when $\theta = 43$ (3 s.f.).

$t = 0$: $18 + A = 90$ so $A = 72$

$t = 8$: $18 + 72e^{-8k} = 54$ so $e^{-8k} = \frac{1}{2}$

$k = \frac{1}{8} \ln 2 = 0.0866$

$72\, e^{-kt} = 25$ so $t = \frac{1}{k} \ln \frac{72}{25}$

$t = 12.2$ minutes

Answer: $A = 72$, $k = 0.0866$, $t = 12.2$ minutes

4 Binomial expansion (fractional or negative powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the binomial expansion of $(4 - x)^{-1}$ in ascending powers of x up to and including the term in x^3.
State the values of x for which it is valid.

$(4 - x)^{-1} = 4^{-1}(1 - \frac{1}{4}x)^{-1} = \frac{1}{4}(1 + u)^{-1}$, $u = -\frac{1}{4}x$.

$(1 + u)^{-1} = 1 + (-1)u + 1u^2 + (-1)u^3 + ...$

Substitute u and multiply by $\frac{1}{4}$.

Valid when $|u| < 1$, i.e. $|x| < 4$.

Answer: $\frac{1}{4} + \frac{1}{16}x + \frac{1}{64}x^2 + \frac{1}{256}x^3 + ...$; $|x| < 4$

5 Coded data: mean, standard deviation, Σx^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 20 observations x, $\Sigma(x - 30) = 70$ and $\Sigma(x - 30)^2 = 325$.
(a) Find the mean of x.
(b) Find the standard deviation of x.
(c) Find the value of Σx^2.

Mean of $(x - 30) = \frac{70}{20} = 3.5$, so mean of $x = 33.5$

Variance $= \frac{325}{20} - (3.5)^2 = 4$

$\sigma = \sqrt{4}$

$\Sigma x^2 = \Sigma(x - 30)^2 + 60\,\Sigma(x - 30) + 20 \times 30^2$

$= 325 + 4200 + 18000$

Answer: (a) 33.5 (b) 2.00 (c) 22525

6 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-10x^2 + 32x + 6}{(x + 3)(x - 3)^2}$ in the form

$\frac{A}{(x + 3)} + \frac{B}{(x - 3)} + \frac{C}{(x - 3)^2}$.

$-10x^2 + 32x + 6 \equiv A(x - 3)^2 + B(x + 3)(x - 3) + C(x + 3)$.

Put $x = -3$: $-180 = A(-6)^2$, so $A = -5$.

Put $x = 3$: $12 = C(6)$, so $C = 2$.

Compare x^2: $-10 = 1A + 1B$, so $B = -5$.

Answer: A = −5, B = −5, C = 2

7 Distance travelled with a change of direction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P moves on a straight line. At time t seconds its velocity is v m s^{-1}, where
$v = 2t^2 - 8t + 6$ for $0 \leq t \leq 5$
Find (a) the times at which P is at instantaneous rest, (b) the total distance travelled by P in the first 5 seconds.

$v = 2(t - 1)(t - 3)$, so $v = 0$ when $t = 1$ and $t = 3$

$v > 0$ for $0 < t < 1$, $v < 0$ for $1 < t < 3$, $v > 0$ for $t > 3$: direction changes at each

$s = \int v\,dt = \frac{2}{3}t^3 - 4t^2 + 6t$

Distances: 0 to 1: $\frac{8}{3}$; 1 to 3: $\frac{8}{3}$; 3 to 5: $\frac{40}{3}$

Total $= \frac{8}{3} + \frac{8}{3} + \frac{40}{3}$

Answer: (a) t = 1, t = 3 (b) $\frac{56}{3}$ m

8 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A rectangle has its base on the x-axis and its upper vertices on
the curve $y = 3 - x^2$, symmetric about the y-axis.
Find the maximum area of the rectangle, with justification.

Base $2x$, height $3 - x^2$: $A = 2x(3 - x^2) = 6x - 2x^3$

$dA/dx = 6 - 6x^2 = 0$, so $x = 1$ $(x > 0)$

$d^2A/dx^2 = -12x < 0$: maximum

$A = 2 \times 1 \times 2$

Answer: 4

9 Force applied then removed (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of mass 10 kg is at rest on rough horizontal ground ($\mu = 0.25$). A horizontal force of 38.5 N acts on it for 7 s and is then removed.

Find (a) the speed when the force is removed, (b) the total distance travelled before the box stops. Take $g = 9.8$ m s^{-2}.

Friction = $\mu mg = 0.25 \times 10 \times 9.8 = 24.5$ N

While pushed: $38.5 - 24.5 = 10a$, so $a = 1.4$

$v = 1.4 \times 7 = 9.8$; $s_1 = \frac{1}{2} \times 1.4 \times 7^2 = 34.3$

After release: deceleration = $\mu g = 2.45$, so $s_2 = \frac{9.8^2}{2 \times 2.45} = 19.6$

Total distance = 34.3 + 19.6

Answer: (a) 9.8 m s^{-1} (b) 53.9 m

10 Arithmetic terms forming a geometric sequence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first term of an arithmetic sequence is 3 and its common difference d is non-zero. The 1st, 3rd and 13th terms are the first three terms of a geometric sequence. Find d and the common ratio.

Terms: 3, 3 + 2d, 3 + 12d.

Geometric: $(3 + 2d)^2 = (3)(3 + 12d)$.

Expand: $4d^2 + 12d + 9 = 36d + 9$, so $d(4d - 24) = 0$.

$d \neq 0$, so $d = 6$. Ratio = $\frac{15}{3} = 5$.

Answer: d = 6, ratio = 5