Day 251 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations
$\log_5 x + \log_5 y = 7$
$x - y = 500$

$\log_5(xy) = 7$, so $xy = 5^7 = 78125$
$x = y + 500$: $y^2 + 500y - 78125 = 0$
$(y - 125)(y + 625) = 0$; $y > 0$ so $y = 125$
Then $x = y + 500$
Answer: x = 625, y = 125

2 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:
$x + 3y = 5$
$3x^2 + 3y^2 = 15$

From the first equation $x = 5 - 3y$.
Substitute into the second equation and collect terms:
this gives $30y^2 - 90y + 60 = 0$.
$y = 1$ or $y = 2$, giving $x = 2$ or $x = -1$.
Answer: (x, y) = (2, 1) and (−1, 2)

3 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 400 kg, carrying a passenger of mass 65 kg, is supported by a vertical cable. The lift is moving downwards and decelerating at 1.4 m s^{-2}.
Find (a) the force the floor exerts on the passenger, (b) the tension in the cable. Take g = 9.8 m s^{-2}.

Take the direction of the acceleration as positive: it is upwards
Passenger: $R - 65 \times 9.8 = 65 \times 1.4$
$R = 65 \times (9.8 + 1.4) = 728$
Lift and passenger: $T - 465 \times 9.8 = (400 + 65) \times 1.4$
$T = 465 \times 11.2 = 5208$
Answer: (a) 728 N (b) 5208 N

4 Arithmetic series from a sum formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

An arithmetic sequence has $u_5 = 19$ and $S_{12} = 300$.
(a) Find the first term and common difference.
(b) Find the least n for which $S_n > 347$.

u_5: $a + 4d = 19$. S_{12}: $12(2a + 11d)/2 = 300$, so $12a + 66d = 300$.
Solving: $a = 3$, $d = 4$.
$S_n = \frac{n}{2}(6 + 4n - 4) = \frac{n(4n + 2)}{2}$.
Test: $S_{12} = 300 \leq 347$, $S_{13} = 351 > 347$.
Answer: (a) a = 3, d = 4 (b) n = 13

5 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ab^x$ passes through (1, 10) and (3, 40).
Find the exact values of a and b, and the value of y when x = 5.

$ab^1 = 10$ and $ab^3 = 40$
Divide: $b^2 = 4$ so $b = 2$
$a = 10 \div 2^1 = 5$
$y = 5 \times 2^5$
Answer: a = 5, b = 2, y = 160

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6 Range and greatest height to speed and angle (AQA, Edexcel, OCR, MEI, Eduqas)

A golf ball is hit from a point O on horizontal ground. It lands 96 m from O, and reaches a maximum height of 12 m above the ground.

Find the angle of projection and the speed of projection, to 3 s.f. Take g = 9.8 m s^{-2}.

Vertical: $0 = u_y^2 - 2 \times 9.8 \times 12$, so $u_y = 15.34$

Time of flight = $\frac{2u_y}{9.8}$ = 3.130 s, so $u_x = \frac{96}{3.130}$ = 30.67

$\tan\theta = \frac{u_y}{u_x}$ = 0.5000, so $\theta = 26.6°$

Speed = $\sqrt{(u_x^2 + u_y^2)}$ = 34.3

Answer: θ = 26.6°, speed 34.3 m s^{-1}

7 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ax^n$. The line of best fit for lg y against lg x is lg y = 0.6 + 2.5 lg x.

(a) Find a and n, giving a to 3 significant figures.

(b) Estimate y when x = 10, to 3 significant figures.

$y = ax^n$ gives lg y = lg a + n lg x, so the gradient is n and the intercept is lg a

n = 2.5

lg a = 0.6 so $a = 10^{0.6}$ = 3.981

x = 10: $y = 3.981 \times 10^{2.5}$

= 1260

Answer: (a) a = 3.98, n = 2.5 (b) 1260

8 Rejected batches: binomial inside binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box holds 12 components, each faulty with probability 0.05, independently. A box is rejected if it has at least 2 faulty components.

(a) Find the probability that a box is rejected.

(b) 12 boxes are inspected. Find the probability that at least 3 are rejected.

Let X ~ B(12, 0.05). P(rejected) = $P(X \geq 2) = 1 - P(X \leq 1)$ = 0.1184

Let Y = number of rejected boxes, Y ~ B(12, 0.1184)

$P(Y \geq 3) = 1 - P(Y \leq 2)$ = 0.162

Answer: (a) 0.118 (b) 0.162

9 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Squares of side x cm are cut from the corners of a square sheet of side 36 cm and the sides folded up to make an open box.

Find the value of x that maximises the volume, and that maximum volume.

$V = x(36 - 2x)^2 = 4x^3 - 144x^2 + 1296x$

$dV/dx = 12x^2 - 288x + 1296 = (36 - 2x)(36 - 6x) = 0$

x < 18, so x = 6

$d^2V/dx^2 = 24x - 288$, negative at x = 6: maximum

$V = 6 \times 24^2$

Answer: x = 6, V = 3456 cm^3

10 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $7k = n^2 + 1$ has no integer solutions (k and n integers).

Consider n = 7q + r, with r = 0, 1, ..., 6.

Squaring, $n^2 \equiv r^2 \pmod 7$, so:

$0 \to 0$; $1 \to 1$; $2 \to 4$; $3 \to 2$; $4 \to 2$; $5 \to 4$; $6 \to 1$ ($r \to n^2 \bmod 7$)

So $n^2 + 1 \equiv 1$ or 2 or 3 or 5 (mod 7).

No residue is 0, so $n^2 + 1$ is never a multiple of 7.

Answer: Proved