Day 250 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations
$\lg x - \lg y = 2$
$x + y = 1010$

$\lg(x/y) = 2$, so $x = 10^2 y = 100y$
$100y + y = 1010$, so $101y = 1010$
Answer: $x = 1000$, $y = 10$

2 Parametric to Cartesian (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations
$x = 3t^2$, $y = 3t^3$.
Show that $y^2 = kx^3$ and find k.

$y^2 = 9t^6$.
$x^3 = 27t^6$.
So $y^2 = \frac{9}{27} x^3$.
Answer: $k = \frac{1}{3}$

3 Particle in equilibrium on two strings (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P of weight 80 N hangs in equilibrium from two light strings AP and BP attached to a horizontal ceiling at A and B (see diagram), where $\tan\alpha = \frac{3}{4}$ and $\tan\beta = \frac{4}{3}$.
Find (a) the tension in each string, (b) the greatest weight P could have if neither tension may exceed 84 N.

Horizontally: $T_1 \cos\alpha = T_2 \cos\beta$, so $T_1 \times \frac{4}{5} = T_2 \times \frac{3}{5}$
Vertically: $T_1 \sin\alpha + T_2 \sin\beta = 80$, and $T_2 = \frac{4}{3} T_1$
$T_1(\frac{3}{5} + \frac{4}{3} \times \frac{4}{5}) = 80$, so $T_1 = 48$ and $T_2 = 64$
The tension in BP is the larger: it is 64 when W = 80
Tensions are proportional to W, so $W_{max} = 80 \times \frac{84}{64} = 105$
Answer: (a) AP: 48 N, BP: 64 N (b) 105 N

4 Transforming a graph: images of points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows y = f(x), made of three straight line segments. Find the coordinates of the images of A, B, C and D on the graph of
$y = 2f(-2x - 8) - 1$.

Inside f: replace x by $-2x - 8$, so x-coordinates become $(x + 8) \div (-2)$.
Outside: y-coordinates become $2 \times y - 1$.
A: $(-1, -7)$ B: $(-5, 9)$
C: $(-6, 1)$ D: $(-7, -5)$
Answer: A′$(-1, -7)$, B′$(-5, 9)$, C′$(-6, 1)$, D′$(-7, -5)$

5 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary points of
$y = \frac{x}{x^2 + 9}$,
and determine the nature of each.

$dy/dx = \frac{(x^2 + 9) - 2x^2}{(x^2 + 9)^2} = \frac{9 - x^2}{(x^2 + 9)^2} = 0$ gives $x = \pm 3$
$y = \pm\frac{1}{6}$ at $x = \pm 3$
d^2y/dx^2 is negative at $x = 3$ and positive at $x = -3$
Answer: $(3, \frac{1}{6})$ maximum; $(-3, -\frac{1}{6})$ minimum

6 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 10 pairs of values: $\Sigma x = 90$, $\Sigma y = 550$, $\Sigma x^2 = 870$, $\Sigma xy = 4998$.
(a) Find the equation of the regression line of y on x.
(b) The x values lie between 5 and 14. Estimate y when x = 13 and comment on its reliability.

$S_{xx} = 870 - \frac{90^2}{10} = 60$

$S_{xy} = 4998 - \frac{90 \times 550}{10} = 48$

$b = \frac{S_{xy}}{S_{xx}} = \frac{48}{60} = \frac{4}{5}$

$a = \bar{y} - b\,\bar{x} = 55 - (\frac{4}{5})(9) = 47.8$

$y = 47.8 + 0.8x$
At x = 13: y = 58.2. This is extrapolation (outside the data range), so may be unreliable
Answer: (a) $y = 47.8 + 0.8x$ (b) 58.2, unreliable (extrapolation)

7 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$g(x) = x^2 - 4$. Also $fg(x) = 4x^2 - 7$, where f is
a linear function. (a) Find f(x).
(b) Solve gf(x) = 21.

(a) Let $f(x) = ax + b$. $fg(x) = a(x^2 - 4) + b = 4x^2 - 7$.
So a = 4 and $-16 + b = -7$, giving b = 9.
$f(x) = 4x + 9$.
(b) $gf(x) = (4x + 9)^2 - 4 = 21$.
$(4x + 9)^2 = 25$, so $4x + 9 = \pm 5$.
$x = -\frac{7}{2}$ or $x = -1$.

Answer: (a) $f(x) = 4x + 9$ (b) $x = -\frac{7}{2}$ or $x = -1$

8 Cumulative distribution function (CIE)

The cdf of X is $F(x) = 0$ for $x < 0$, $F(x) = 1 - (1 - \frac{x}{5})^2$ for $0 \le x \le 5$, and $F(x) = 1$ for $x > 5$.
(a) Find the pdf of X.
(b) Find E(X).
(c) Find P(1.25 < X < 2.5) and the median, to 3 significant figures.

(a) $f(x) = F'(x) = \frac{2}{5}(1 - \frac{x}{5})^1$ for $0 \le x \le 5$

(b) $E(X) = \int x\, f(x)\, dx$ from 0 to 5 $= \frac{5}{3}$

(c) $P = F(2.5) - F(1.25) = 0.3125$
Median m: $F(m) = 1/2$, so $m = 5(1 - (1/2)^{1/2})$
m = 1.464

Answer: (a) $f(x) = \frac{2}{5}(1 - \frac{x}{5})^1$ (b) $\frac{5}{3}$ (c) 0.312, median 1.46

9 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A rectangle has its base on the x-axis and its upper vertices on
the curve $y = 48 - x^2$, symmetric about the y-axis.
Find the maximum area of the rectangle, with justification.

Base 2x, height $48 - x^2$: $A = 2x(48 - x^2) = 96x - 2x^3$
$dA/dx = 96 - 6x^2 = 0$, so x = 4 (x > 0)
$d^2A/dx^2 = -12x < 0$: maximum
$A = 2 \times 4 \times 32$
Answer: 256

10 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time t = 0 hours, A is at ($\mathbf{i} + 7\mathbf{j}$) km relative to a fixed origin and B is at ($-7\mathbf{i} + 26\mathbf{j}$) km. A has velocity ($3\mathbf{i} - 4\mathbf{j}$) km h^{-1} and B has velocity ($6\mathbf{i} - 8\mathbf{j}$) km h^{-1}.
Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (-8\mathbf{i} + 19\mathbf{j}) + t(3\mathbf{i} - 4\mathbf{j})$

$|\mathbf{d}|^2 = (3t - 8)^2 + (-4t + 19)^2 = 25t^2 - 200t + 425$

Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\frac{-100}{25} = 4$

$\mathbf{d} = (-8\mathbf{i} + 19\mathbf{j}) + 4(3\mathbf{i} - 4\mathbf{j}) = (4\mathbf{i} + 3\mathbf{j})$, so least distance = 5

Answer: (a) t = 4 h (b) 5 km