Day 25 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos(3\theta - 10°) = -\frac{1}{2}$

for $0° \leq \theta \leq 360°$.

Let $u = 3\theta - 10°$; then $\cos u = -\frac{1}{2}$

Base solutions: $u = 120, -120°$ (+ multiples of 360°)

$\theta = (u + 10°) \div 3$, keeping those in range

Answer: θ = 43.3°, 83.3°, 163.3°, 203.3°, 283.3°, 323.3°

2 Equations in disguise (substitution) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x - 6\sqrt{x} + 8 = 0$.

Let $u = \sqrt{x}$: $u^2 - 6u + 8 = 0$.

$(u - 4)(u - 2) = 0$, so $\sqrt{x} = 4$ or $\sqrt{x} = 2$.

Square: $x = 16$ or $x = 4$.

Answer: x = 4 or x = 16

3 Poisson: scaling the rate, then binomial (CIE)

Calls arrive at a switchboard at an average rate of 6 per hour, independently and at random.

(a) Find the probability of at least 3 in a 15-minute period.

(b) Find the probability that, of 6 separate 15-minute periods, exactly 2 have no arrivals.

15-minute period: $\lambda = 6 \times \frac{15}{60} = 1.5$, so $X \sim Po(1.5)$

(a) $P(X \geq 3) = 1 - P(X \leq 2) = 0.1912$

(b) $P(X = 0) = e^{-1.5} = 0.2231$

$Y \sim B(6, 0.2231)$: $P(Y = 2) = 0.272$

Answer: (a) 0.191 (b) 0.272

4 Transformations: new equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = x^2 - 2x + 5$ is translated by $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

and then a stretch, scale factor 3, parallel to the x-axis.

Find the equation of the resulting curve in the form $y = ax^2 + bx + c$.

Translation by $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$: $y = f(x - 2) + 3 = x^2 - 6x + 16$.

Then a stretch, scale factor 3, parallel to the x-axis: replace x by $\frac{1}{3}x$: $y = f(\frac{1}{3}x - 2) + 3$.

Expand and simplify to give the quadratic.

Answer: $y = \frac{1}{9}x^2 - 2x + 16$

5 Range and greatest height to speed and angle (AQA, Edexcel, OCR, MEI, Eduqas)

A golf ball is hit from a point O on horizontal ground. It lands 80 m from O, and reaches a maximum height of 40 m above the ground.

Find the angle of projection and the speed of projection, to 3 s.f. Take $g = 9.8$ m s^{-2}.

Vertical: $0 = u_y^2 - 2 \times 9.8 \times 40$, so $u_y = 28.00$

Time of flight = $\frac{2u_y}{9.8} = 5.714$ s, so $u_x = \frac{80}{5.714} = 14.00$

$\tan\theta = \frac{u_y}{u_x} = 2.000$, so $\theta = 63.4°$

Speed = $\sqrt{(u_x^2 + u_y^2)} = 31.3$

Answer: θ = 63.4°, speed 31.3 m s^{-1}

6 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$\frac{dy}{dx} = y(4 - y)$, and y = 2 when x = 0.

Solve to find the value of x when y = 3.

$\frac{1}{y(4-y)} \equiv \frac{1}{4}(\frac{1}{y} + \frac{1}{4-y})$

$\frac{1}{4}\ln|\frac{y}{4-y}| = x + c$; x = 0, y = $\frac{4}{2}$ gives c = 0

$\frac{y}{4-y} = e^{4x}$; with y = 3: $3 = e^{4x}$

Answer: x = $\frac{\ln 3}{4}$

7 Discrete random variable with a probability formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$P(X = x) = kx^2$ for x = 1, 2, ..., 4, and 0 otherwise.

(a) Show that k = $\frac{1}{30}$.

(b) Find E(X) and Var(X).

(c) Find Var(3X + 1).

(a) $\Sigma P(X = x) = k(1 + 4 + 9 + 16) = 30k = 1$, so k = $\frac{1}{30}$

(b) $E(X) = \Sigma x P(X = x) = \frac{1\times1 + 2\times4 + 3\times9 + 4\times16}{30} = \frac{10}{3}$

$E(X^2) = \frac{1\times1 + 4\times4 + 9\times9 + 16\times16}{30} = \frac{59}{5}$

$Var(X) = \frac{59}{5} - \frac{10}{3}^2 = \frac{31}{45}$

(c) $Var(3X + 1) = 3^2 Var(X) = 9 \times \frac{31}{45} = \frac{31}{5}$

Answer: (b) E(X) = $\frac{10}{3}$, Var(X) = $\frac{31}{45}$ (c) $\frac{31}{5}$

8 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (−1, 3, 4), (5, 5, −3), (0, −1, 5).

Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix}-6\\-2\\7\end{pmatrix}$, $BC = \begin{pmatrix}-5\\-6\\8\end{pmatrix}$; $BA \cdot BC = 98$

$|BA| = \sqrt{89}$, $|BC| = \sqrt{125}$

$\cos ABC = \frac{98}{\sqrt{89}\sqrt{125}} = 0.9291$, so ABC = 21.7°

$|BA \times BC| = |\begin{pmatrix}26\\13\\26\end{pmatrix}| = \sqrt{1521}$

Area = $\frac{1}{2}|BA \times BC|$

Answer: ABC = 21.7°, area = $\frac{39}{2}$

9 Non-uniform rod: centre of mass and tilting (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB of length 8 m and mass 10 kg rests on supports at C and D, where AC = 2 m and AD = 7 m, with R_C = 1.5 R_D.

Find (a) the distance of the centre of mass from A. A particle of mass m kg at B puts the rod on the point of tilting about D. (b) Find m.

R_C = 1.5 R_D. Moments about A: R_C × 2 + R_D × 7 = 10 g × AG

R_D(1.5 × 2 + 7) = 10 g × AG with R_D(1 + 1.5) = 10 g, so AG = $\frac{1.5 \times 2 + 7}{2.5} = 4$

Tilting about D: R_C = 0. Moments about D: m g (8 − 7) = 10 g (7 − 4)

m = $\frac{10 \times 3}{1} = 30$

Answer: (a) 4 m (b) 30

10 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

f(x) = $\frac{21x - 1}{(1 + 3x)(1 - 3x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) 21x − 1 ≡ A(1 − 3x) + B(1 + 3x): A = −4, B = 3.

f(x) = $\frac{-4}{1 + 3x} + \frac{3}{1 - 3x}$.

(b) $\frac{1}{1 + cx}$ = 1 − cx + c^2x^2 − c^3x^3 + ... (use c = 3, then c = −3).

Coefficients of x^j: $-4(-3)^j + 3(3)^j$.

Valid for |x| < $\frac{1}{3}$ (both expansions).

Answer: (a) A = −4, B = 3 (b) −1 + 21x − $9x^2$ + $189x^3$ + ...; |x| < $\frac{1}{3}$