Day 249 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Stationary points of trigonometric curves (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary points of
$y = 2x - 4\sin x$ for $0 \le x \le 2\pi$,
and determine their nature.

$dy/dx = 2 - 4\cos x = 0$, so $\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$

$d^2y/dx^2 = 4\sin x$
Its sign at each point decides maximum or minimum

Answer: $(\frac{\pi}{3}, \frac{2}{3}\pi - 2\sqrt{3})$ minimum; $(\frac{5\pi}{3}, \frac{10}{3}\pi + 2\sqrt{3})$ maximum

2 Projectile through a given point: two angles (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 28 m s^{-1}, at angle θ above the horizontal. It passes through the point P, 44 m horizontally and 27 m vertically from O.
Find the two possible values of $\tan\theta$, and the two possible values of θ to 1 d.p. Take $g = 9.8$ m s^{-2}.

Horizontal: $x = 28\cos\theta \times t$. Vertical: $y = 28\sin\theta \times t - 4.9t^2$

Eliminating t: $y = x\tan\theta - \frac{9.8x^2}{2u^2}(1 + \tan^2\theta)$

$\frac{9.8 \times 44^2}{2 \times 28^2} = 12.1$, so $27 = 44\tan\theta - 12.1(1 + \tan^2\theta)$

$12.1\tan^2\theta - 44\tan\theta + 39.1 = 0$

$\tan\theta = \frac{17}{11}$ or $\frac{23}{11}$, so $\theta = 57.1°$ or $64.4°$

Answer: $\tan\theta = \frac{17}{11}$ or $\frac{23}{11}$; $\theta = 57.1°$ or $64.4°$

3 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-2 + \sqrt{3}}{1 - \sqrt{3}}$ in the form $p + q\sqrt{3}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(1 + \sqrt{3})$.
Numerator: $(-2 + \sqrt{3})(1 + \sqrt{3}) = 1 - \sqrt{3}$.
Denominator: $1^2 - 1^2(3) = -2$.

So the fraction = $\frac{1 - \sqrt{3}}{-2}$.

Answer: $-\frac{1}{2} + \frac{1}{2}\sqrt{3}$

4 Compound change with logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drug dose of 20 mg decays by 10% each hour.
Find the first whole number of hours after which less than 1 mg remains.

$20 \times 0.90^n < 1$, so $0.90^n < \frac{1}{20}$

$n > \frac{\ln\frac{1}{20}}{\ln 0.90} = 28.43$ ($\ln 0.90$ is negative, so the inequality reverses)
$n = 29$
Answer: 29

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5 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is dropped from a height of 4 m. After each bounce it rises to $\frac{2}{3}$ of its previous height.

(a) Find the total distance it travels before it stops.
(b) After how many bounces does it rise less than 0.1 m?

(a) Down H, then up and down each time: $H + 2Hr + 2Hr^2 + ...$

$= H + \frac{2Hr}{1 - r} = 4 + \frac{2 \times 4 \times \frac{2}{3}}{1 - \frac{2}{3}} = 20.$

(b) Rebound height after n bounces $= 4(\frac{2}{3})^n < 0.1.$

$n > \frac{\log(0.025)}{\log(0.666667)} = 9.10$, so n = 10.

Answer: (a) 20 m (b) 10 bounces

6 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ax^n$. The line of best fit for lg y against lg x is $\lg y = 0.25 - 2 \lg x$.
(a) Find a and n, giving a to 3 significant figures.
(b) Estimate y when x = 20, to 3 significant figures.

$y = ax^n$ gives $\lg y = \lg a + n \lg x$, so the gradient is n and the intercept is lg a
$n = -2$
$\lg a = 0.25$ so $a = 10^{0.25} = 1.778$
$x = 20$: $y = 1.778 \times 20^{-2}$
$= 0.00445$
Answer: (a) a = 1.78, n = −2 (b) 0.00445

7 Stationary points of x^n times a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 2x^2\sqrt{45 - x}$, for $0 < x < 45$, has one stationary point. Find its exact coordinates and determine its nature.

$dy/dx = \frac{2x(180 - 5x)}{2\sqrt{45 - x}}$
$dy/dx = 0$: $180 - 5x = 0$, so $x = 36$
$y = 2 \times 36^2 \times 3$
Gradient changes from positive to negative: a maximum
Answer: (36, 7776), maximum

8 Transformations of y = x^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Describe a sequence of two transformations that maps the curve $y = x^2$ onto the curve $y = 5x^2 + 40x + 73$.

Complete the square: $y = 5(x + 4)^2 - 7$.
Step 1: $y = x^2 \rightarrow y = 5x^2$: a stretch, scale factor 5, parallel to the y-axis.
Step 2: replace x by (x + 4) and add −7: translation by $\begin{pmatrix} -4 \\ -7 \end{pmatrix}$.
Answer: a stretch, scale factor 5, parallel to the y-axis, then translation $\begin{pmatrix} \mathbf{-4} \\ \mathbf{-7} \end{pmatrix}$

9 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 2 kg lies on a smooth horizontal table, with a block of mass 4 kg on it ($\mu = 0.3$ between them). A horizontal force P N acts on the plank.

(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 52.92. Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{6}$. The block is accelerated only by friction: $F = 4a \leq \mu mg$

So $a \leq \mu g = 2.94$, and $P_{max} = 6 \times 2.94 = 17.64$

$P = 52.92 > 17.64$, so the block slips. Block: $a = \mu g = 2.94$

Plank: $52.92 - 11.76 = 2a$, so $a = 20.58$

Answer: (a) 17.64 N (b) block 2.94 m s^{-2}, plank 20.58 m s^{-2}

10 Rejected batches: binomial inside binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box holds 11 components, each faulty with probability 0.05, independently. A box is rejected if it has at least 2 faulty components.

(a) Find the probability that a box is rejected.

(b) 13 boxes are inspected. Find the probability that at least 3 are rejected.

Let $X \sim B(11, 0.05)$. $P(\text{rejected}) = P(X \geq 2) = 1 - P(X \leq 1) = 0.1019$

Let Y = number of rejected boxes, $Y \sim B(13, 0.1019)$

$P(Y \geq 3) = 1 - P(Y \leq 2) = 0.140$

Answer: (a) 0.102 (b) 0.140