Day 247 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Integrals of the form f'(x)/f(x) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$$\int_0^1 \frac{10x + 20}{x^2 + 4x + 6}\,dx$$

Numerator = 5 × derivative of denominator

Integral = $[5 \ln(x^2 + 4x + 6)]$

$= 5 (\ln 11 - \ln 6)$

Answer: $5\ln 11 - 5\ln 2 - 5\ln 3$

2 Coded data: mean, standard deviation, Σx² (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 25 observations x, $\Sigma(x - 50) = 40$ and $\Sigma(x - 50)^2 = 489$.

(a) Find the mean of x.

(b) Find the standard deviation of x.

(c) Find the value of Σx^2.

Mean of $(x - 50) = \frac{40}{25} = 1.6$, so mean of x = 51.6

Variance $= \frac{489}{25} - (1.6)^2 = 17$

$\sigma = \sqrt{17}$

$\Sigma x^2 = \Sigma(x - 50)^2 + 100\,\Sigma(x - 50) + 25 \times 50^2$

$= 489 + 4000 + 62500$

Answer: (a) 51.6 (b) 4.12 (c) 66989

3 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (4 kg) lies on a rough table with μ = 0.3, and is joined by a light string over a smooth pulley (see diagram) to particle B (4 kg) on a smooth slope, α = 30°.

Released from rest, the string is cut when B has moved 3.75 m. Find the further distance A slides before stopping.

A: $T - 0.3 \times 4 \times 9.8 = 4a$. B: $4 \times 9.8 \times \frac{1}{2} - T = 4a$

Adding: $9.8(4 \times \frac{1}{2} - 0.3 \times 4) = 8a$, so $a = 0.98$

$T = 4(0.98 + 0.3 \times 9.8) = 15.68$

When cut: $v^2 = 2 \times 0.98 \times 3.75 = 7.35$. Then A decelerates at $\mu g = 2.94$

Further distance $= \frac{7.35}{2 \times 2.94} = 1.25$

Answer: 1.25 m

4 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation $y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes x = 1 and y = 3, so p = −1 and q = 3.

$y = \frac{a}{x - 1} + 3$.

P(2, 9): $9 = \frac{a}{1} + 3$, so a = 6.

Answer: a = 6, p = −1, q = 3

5 Tangent and axes triangle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tangent to $y = \frac{24}{x^2 + 8}$ at the point where $x = 1$ meets the axes at A and B. Find the exact area of triangle OAB, where O is the origin.

At $x = 1$: $y = \frac{8}{3}$. Differentiate: gradient $= -\frac{16}{27}$

Tangent: $y - \frac{8}{3} = (-\frac{16}{27})(x - 1)$, so $y = -\frac{16}{27}x + \frac{88}{27}$

y-intercept $\frac{88}{27}$; x-intercept $\frac{11}{2}$

Area $= \frac{1}{2} \times |\frac{88}{27}| \times |\frac{11}{2}|$

Answer: $\frac{242}{27}$

6 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(25 - x)^{\frac{1}{2}}$ up to and including the term in x^2.

(b) Use $x = 0.02$ to estimate $\sqrt{24.98}$ to 5 decimal places.

(a) $(25 - x)^{\frac{1}{2}} = 5(1 - \frac{1}{25}x)^{\frac{1}{2}}$.

$= 5 - \frac{1}{10}x - \frac{1}{1000}x^2 + ...$

(b) $25 - x$ with $x = 0.02$ gives 24.98.

Substitute: $5 + (-\frac{1}{10})(0.02) + (-\frac{1}{1000})(0.02)^2$.

$= 4.99800$ (5 d.p.).

Answer: (a) $\mathbf{5 - \frac{1}{10}x - \frac{1}{1000}x^2}$ **(b) 4.99800**

7 Displacement of the form R sin(ωt + φ) (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a line. Its displacement from a fixed point O at time t seconds is s metres, where $s = 5 \sin t + 12 \cos t$.

Find (a) the greatest distance of P from O, (b) the greatest speed of P, (c) the first time P is at its greatest distance from O (3 s.f.).

Write $s = R \sin(t + \varphi)$ with $R \cos \varphi = 5$ and $R \sin \varphi = 12$

$R = \sqrt{(5^2 + 12^2)} = 13$ and $\tan \varphi = \frac{12}{5}$, so $\varphi = 1.176$ rad

Greatest distance = R = 13

$v = \frac{ds}{dt} = 13 \times 1 \cos(t + \varphi)$, so greatest speed = 13

s is greatest when $t + \varphi = \frac{\pi}{2}$: $t = (1.5708 - 1.176) \div 1$

Answer: (a) 13 m (b) 13 m s^{-1} (c) 0.395 s

8 Sums of inverse tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B are acute angles with $\tan A = \frac{1}{6}$ and $\tan B = \frac{5}{7}$.

Find $\tan(A + B)$ and hence show that $A + B = \frac{\pi}{4}$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{6} + \frac{5}{7}}{1 - \frac{5}{42}} = \frac{\frac{37}{42}}{\frac{37}{42}} = 1$$

A + B is acute, so $A + B = \pi/4$

Answer: tan(A + B) = 1, A + B = $\mathbf{\frac{\pi}{4}}$

9 Binomial: finding p from a ratio (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(11, p)$. It is given that $P(X = 2) = \frac{5}{3} \times P(X = 1)$.

(a) Find the value of p.
(b) Find $P(X = 0)$, correct to 3 significant figures.

$\frac{110}{2} p^2(1 - p)^9 = \frac{5}{3} \times 11\, p(1 - p)^{10}$

Divide by $p(1 - p)^9$: $5\, p = \frac{5}{3}(1 - p)$

$(5 + \frac{5}{3})p = \frac{5}{3}$

$p = \frac{1}{4}$

$P(X = 0) = (1 - p)^{11} = (\frac{3}{4})^{11} = 0.0422$

Answer: (a) $p = \frac{1}{4}$ (b) 0.0422

10 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-14x^2 - 18x + 31}{(x + 4)(2x - 3)^2}$ in the form

$\frac{A}{(x + 4)} + \frac{B}{(2x - 3)} + \frac{C}{(2x - 3)^2}$.

$-14x^2 - 18x + 31 \equiv A(2x - 3)^2 + B(x + 4)(2x - 3) + C(x + 4)$.

Put $x = -4$: $-121 = A(-11)^2$, so $A = -1$.

Put $x = \frac{3}{2}$: $-\frac{55}{2} = C(\frac{11}{2})$, so $C = -5$.

Compare x^2: $-14 = 4A + 2B$, so $B = -5$.

Answer: A = −1, B = −5, C = −5