Day 246 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Proof by contradiction (logarithms) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\log_3 10$ is irrational.

Assume $\log_3 10 = \frac{m}{n}$, m, n positive integers.

Then $3^{\frac{m}{n}} = 10$, so $3^m = 10^n$.

Prime factors of 3^m and 10^n must be identical,

but 2 divides 10^n but 2 does not divide 3^m.

This is a contradiction, so $\log_3 10$ is irrational.

Answer: Proved

2 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population of bacteria is modelled by $P = P_0e^{kt}$, t in hours.

Initially P = 50 and after 4 hours $P = \frac{125}{2}$.

Find k, and the time for the population to reach 500, to 3 s.f.

$P_0 = 50$. $50\ e^{4k} = \frac{125}{2}$ so $e^{4k} = \frac{5}{4}$

$k = \frac{1}{4} \ln \frac{5}{4} = 0.0558$

$50\ e^{kt} = 500$ so $e^{kt} = 10$, $t = \frac{1}{k} \ln 10$

t = 41.3 hours

Answer: k = 0.0558, t = 41.3 hours

3 Replacing an observation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

25 recorded times (seconds) have mean 39 and standard deviation 5.
One value was recorded as 36 but should have been 21.
Find the correct mean and the correct standard deviation.

$\Sigma x = 25 \times 39 = 975$, so corrected $\Sigma x = 975 - 36 + 21 = 960$

Correct mean $= \frac{960}{25} = 38.40$

$\Sigma x^2 = 25(25 + 1521) = 38650$, corrected $\Sigma x^2 = 38650 - 1296 + 441 = 37795$

Variance $= \frac{37795}{25} - 38.400^2 = 37.240$, $\sigma = 6.102$

Answer: mean 38.4, standard deviation 6.10

4 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A pile of logs has 17 logs in its bottom row. Each row above has 1 fewer logs than the row below, and the pile has as many rows as possible, each with at least one log. Find the number of rows and the total number of logs.

Row n has $17 - 1(n - 1) = -n + 18$ logs; need this ≥ 1.
$1(n - 1) \leq 16$, so $n \leq 17$: 17 rows.

Top row = 1 logs. $S = \frac{17}{2}(17 + 1) = 153$.

Answer: 17 rows, 153 logs

5 Distance travelled with a change of direction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P moves on a straight line. At time t seconds its velocity is v m s^{-1}, where
$v = t^2 - 4t + 3$ for $0 \leq t \leq 4$
Find (a) the times at which P is at instantaneous rest, (b) the total distance travelled by P in the first 4 seconds.

$v = (t - 1)(t - 3)$, so $v = 0$ when $t = 1$ and $t = 3$
$v > 0$ for $0 < t < 1$, $v < 0$ for $1 < t < 3$, $v > 0$ for $t > 3$: direction changes at each

$s = \int v\,dt = \frac{1}{3}t^3 - 2t^2 + 3t$

Distances: 0 to 1: $\frac{4}{3}$; 1 to 3: $\frac{4}{3}$; 3 to 4: $\frac{4}{3}$

Total = $\frac{4}{3} + \frac{4}{3} + \frac{4}{3}$

Answer: (a) t = 1, t = 3 (b) 4 m

6 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ax^n$. The line of best fit for lg y against lg x is $\lg y = 0.25 - 0.5 \lg x$.
(a) Find a and n, giving a to 3 significant figures.
(b) Estimate y when x = 100, to 3 significant figures.

$y = ax^n$ gives $\lg y = \lg a + n \lg x$, so the gradient is n and the intercept is lg a
$n = -0.5$
$\lg a = 0.25$ so $a = 10^{0.25} = 1.778$
$x = 100$: $y = 1.778 \times 100^{-0.5}$
$= 0.178$
Answer: (a) a = 1.78, n = −0.5 (b) 0.178

7 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (10 kg) lies on a rough table with $\mu = 0.25$, and is joined by a light string over a smooth pulley (see diagram) to particle B (10 kg) on a smooth slope, $\sin\alpha = \frac{4}{5}$.
Released from rest, the string is cut when B has moved 1.5 m. Find the further distance A slides before stopping.

A: $T - 0.25 \times 10 \times 9.8 = 10a$. B: $10 \times 9.8 \times \frac{4}{5} - T = 10a$

Adding: $9.8(10 \times \frac{4}{5} - 0.25 \times 10) = 20a$, so $a = 2.695$

$T = 10(2.695 + 0.25 \times 9.8) = 51.45$
When cut: $v^2 = 2 \times 2.695 \times 1.5 = 8.085$. Then A decelerates at $\mu g = 2.45$
Further distance = $\frac{8.085}{2 \times 2.45} = 1.65$

Answer: 1.65 m

8 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 + xy + 4y^2 = 135$.
Find the coordinates of the points on C where the tangent is parallel to the x-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + y}{x + 8y}$

Horizontal tangent when $2x + y = 0$, so $y = -\frac{2x}{1}$

Substitute into the equation of C: $x^2 + (xy) + (4y^2) = 135$
Solve for the remaining coordinate
Answer: (3, −6) and (−3, 6)

9 Parametric to Cartesian (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations
$x = -3 + 4\cos t$, $y = 1 + 4\sin t$.
Find its Cartesian equation.

$\cos t = \frac{x + 3}{4}$, $\sin t = \frac{y - 1}{4}$.

Use $\cos^2 t + \sin^2 t = 1$:

$(x + 3)^2 + (y - 1)^2 = 16$.

Answer: $(x + 3)^2 + (y - 1)^2 = 16$

10 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix} + s\begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $-1 - s = -1 + 2t$; $7 + 3s = -1 + 2t$; $-3 - 3s = 6 - 3t$
Solving the first two: $s = -2$, $t = 1$; the third equation is satisfied
Intersection: (1, 1, 3)

$\cos\varphi = \frac{|13|}{\sqrt{19}\sqrt{17}} = 0.7233$, $\varphi = 43.7°$

Answer: Lines meet at (1, 1, 3); angle 43.7°