Day 245 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $5^{2x} - 27 \times 5^x + 50 = 0$,
giving each answer exactly and, where not an integer, to 3 s.f.

Let $u = 5^x$: $u^2 - 27u + 50 = 0$
$(u - 25)(u - 2) = 0$, so $5^x = 25$ or $5^x = 2$
Take logarithms: $x = \log_5 2 = 0.431$; $x = \log_5 25 = 2$
Answer: $x = \log_5 2 = 0.431$; $x = \log_5 25 = 2$

2 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car P passes a point O travelling at a constant speed of 16 m s^{-1}. 5 s later, car Q leaves O from rest with constant acceleration 2 m s^{-2}, travelling in the same direction.
Q overtakes P at the point X. Find (a) the time Q takes to reach X, (b) the distance OX, (c) the speed of Q at X.

Let Q travel for t s. P: distance = $16(t + 5)$; Q: distance = $\frac{1}{2} \times 2 \times t^2$
Equate: $2t^2 - 32t - 160 = 0$, so $(t - 20)(2t + 8) = 0$
$t = 20$ (the other root is negative)
OX = $2 \times 20^2 / 2 = 400$ m, and speed = $2 \times 20 = 40$
Answer: (a) 20 s (b) 400 m (c) 40 m s^{-1}

3 Partial fractions (improper) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{x^3 - 2x^2 - 9x + 16}{(x - 4)(x - 2)}$ in the form $ax + b + \frac{C}{(x - 4)} + \frac{D}{(x - 2)}$.

Divide: the numerator has degree 3, the denominator degree 2.
$(x^3 - 2x^2 - 9x + 16) \div (x^2 - 6x + 8)$ gives quotient $x + 4$.
Remainder: $7x - 16$.
Put $x = 4$: remainder = $C(2)$ so $C = 6$. Put $x = 2$: $D = 1$.
Answer: $x + 4 + \frac{6}{(x - 4)} + \frac{1}{(x - 2)}$

4 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = x^3 + x^2 - 6x - 4$.
(a) Show that $f(x) = 0$ has a root α with $-1 < \alpha < 0$.
(b) Use Newton-Raphson with $x_0 = 0$ to find x_1 and x_2
to 4 decimal places.

$f(-1) = 2$ and $f(0) = -4$: a change of sign, f continuous
$f'(x) = 3x^2 + 2x - 6$, so $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
$x_1 = 0 - \frac{-4.000}{-6.000} = -0.6667$
$x_2 = -0.6667 - \frac{0.1481}{-6.000} = -0.6420$
Answer: $x_1 = -0.6667$, $x_2 = -0.6420$

5 PMCC from summary data and a hypothesis test (AQA, Edexcel, OCR, MEI, Eduqas)

For 12 pairs of values, $S_{xx} = 324$, $S_{yy} = 144$ and $S_{xy} = -131$.
(a) Calculate the product moment correlation coefficient r.
(b) The critical value for a one-tailed test at the 5% level with n = 12 is 0.4973. Carry out the test of H_0: $\rho = 0$ against H_1: $\rho < 0$.

$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}} = \frac{-131}{\sqrt{(324 \times 144)}} = \frac{-131}{216} = -0.6065$
H_0: $\rho = 0$, H_1: $\rho < 0$
$|r| = 0.6065 >$ the critical value 0.4973
Reject H_0: there is sufficient evidence of negative correlation between the variables
Answer: (a) $r = -0.606$ (b) reject H_0

6 Tangent and axes triangle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tangent to $y = \frac{18}{x^2 + 5}$ at the point where $x = 2$ meets the axes at A and B. Find the exact area of triangle OAB, where O is the origin.

At $x = 2$: $y = 2$. Differentiate: gradient $= -\frac{8}{9}$

Tangent: $y - 2 = (-\frac{8}{9})(x - 2)$, so $y = -\frac{8}{9}x + \frac{34}{9}$

y-intercept $\frac{34}{9}$; x-intercept $\frac{17}{4}$

Area $= \frac{1}{2} \times |\frac{34}{9}| \times |\frac{17}{4}|$

Answer: $\frac{289}{36}$

7 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{6}{2x^2 - 4x + 6}$ for all real x.

Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$2x^2 - 4x + 6 = 2(x - 1)^2 + 4$.

$(x - 1)^2 \geq 0$, so the denominator is at least 4, with equality at $x = 1$.

The denominator is always positive, so $f(x) > 0$.

Greatest $f = \frac{6}{4} = \frac{3}{2}$ at $x = 1$.

Answer: greatest value $\frac{3}{2}$ at $x = 1$; range $0 < f(x) \leq \frac{3}{2}$

8 Displacement of the form R sin(ωt + φ) (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a line. Its displacement from a fixed point O at time t seconds is s metres, where $s = 5 \sin \frac{t}{2} + 12 \cos \frac{t}{2}$.

Find (a) the greatest distance of P from O, (b) the greatest speed of P, (c) the first time P is at its greatest distance from O (3 s.f.).

Write $s = R \sin(\frac{t}{2} + \varphi)$ with $R \cos \varphi = 5$ and $R \sin \varphi = 12$

$R = \sqrt{(5^2 + 12^2)} = 13$ and $\tan \varphi = \frac{12}{5}$, so $\varphi = 1.176$ rad

Greatest distance $= R = 13$

$v = \frac{ds}{dt} = 13 \times \frac{1}{2} \cos(\frac{t}{2} + \varphi)$, so greatest speed $= \frac{13}{2}$

s is greatest when $\frac{t}{2} + \varphi = \frac{\pi}{2}$: $t = (1.5708 - 1.176) \div \frac{1}{2}$

Answer: (a) 13 m (b) $\frac{13}{2}$ m s^{-1} (c) 0.790 s

9 Conditional probability with a normal variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A student's time on a task, in minutes, is $N(39, 12^2)$.

Given that it exceeds 27, find the probability that it exceeds 36.

Required $= P(X > 36 \mid X > 27) = \frac{P(X > 36)}{P(X > 27)}$

$P(X > 36) = P(Z > -0.250) = 0.5987$

$P(X > 27) = P(Z > -1.000) = 0.8413$

$= \frac{0.5987}{0.8413} = 0.712$

Answer: 0.712

10 Perpendicular lines and area (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the diagram, line l_1 passes through A and B.
Line l_2 is perpendicular to l_1 at B and meets the
x-axis at C. Find the coordinates of C and the area of triangle ABC.

Gradient of $l_1 = \frac{6-4}{-2-4} = -\frac{1}{3}$.
Gradient of $l_2 = 3$. Equation: $y - 6 = 3(x + 2)$.
$y = 0$: $x = -4$, so C$(-4, 0)$.
AB $= 2\sqrt{10}$, BC $= 2\sqrt{10}$: area $= \frac{1}{2} \times$ AB $\times$ BC.
Answer: C$(-4, 0)$; area 20