Day 244 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{-5x + 1}{(1 - 3x)(1 - x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $-5x + 1 \equiv A(1 - x) + B(1 - 3x)$: $A = -1$, $B = 2$.

$f(x) = \frac{-1}{1 - 3x} + \frac{2}{1 - x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use $c = -3$, then $c = -1$).

Coefficients of x^j: $-1(3)^j + 2(1)^j$.

Valid for $|x| < \frac{1}{3}$ (both expansions).

Answer: (a) $A = -1$, $B = 2$ (b) $1 - x - 7x^2 - 25x^3 + \ldots$; $|x| < \frac{1}{3}$

2 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 6 strips to estimate

$\int_0^1 \ln(1 + x^2)\,dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

$h = \frac{1}{6}$; ordinates at $x = 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \ldots, 1$

Sum $= \frac{h}{2}[y_0 + y_6 + 2(y_1 + \ldots + y_5)] = 0.2663$

$f''(x) > 0$ on the interval, so the curve is convex and the trapezia lie above the curve

Answer: 0.2663; overestimate

3 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

An arrow is projected from a point O on horizontal ground with speed 49 m s^{-1} at an angle α above the horizontal, where $\tan\alpha = \frac{4}{3}$. Find the horizontal distance between the two points at which the arrow is 58.8 m above the ground. Take g = 9.8 m s^{-2}.

Vertical: $58.8 = 39.2\,t - 4.9t^2$, i.e. $4.9t^2 - 39.2t + 58.8 = 0$

$t = \frac{39.2 \pm \sqrt{(39.2^2 - 19.6 \times 58.8)}}{9.8} = \frac{39.2 \pm 19.6}{9.8}$

The two times differ by 4 s (symmetry about the top of the path)

Horizontal distance = 29.4×4

Answer: 117.6 m

4 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + 3n + 3$, $u_1 = -4$.

(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.

(b) Find the least n for which $u_n > 713$.

$u_n = u_1 + \sum_{k=1}^{n-1}(3k + 3) = -4 + 3 \times \frac{(n - 1)n}{2} + 3(n - 1)$.

$= \frac{3}{2}n^2 + \frac{3}{2}n - 7$.

So $a = \frac{3}{2}$, $b = \frac{3}{2}$, $c = -7$.

$u_{21} = 686 \leq 713$ and $u_{22} = 752 > 713$.

Answer: (a) $a = \frac{3}{2}$, $b = \frac{3}{2}$, $c = -7$ (b) $n = 22$

Day 244 **A Level Maths: S Tier 10-a-day: Model Solutions**

5 Replacing an observation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

12 recorded times (seconds) have mean 23 and standard deviation 3.
One value was recorded as 14 but should have been 5.
Find the correct mean and the correct standard deviation.

$\Sigma x = 12 \times 23 = 276$, so corrected $\Sigma x = 276 - 14 + 5 = 267$

Correct mean $= \frac{267}{12} = 22.25$

$\Sigma x^2 = 12(9 + 529) = 6456$, corrected $\Sigma x^2 = 6456 - 196 + 25 = 6285$

Variance $= \frac{6285}{12} - 22.250^2 = 28.688$, $\sigma = 5.356$

Answer: mean 22.2, standard deviation 5.36

6 Compound change with logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A colony of 200 cells grows by 5% each day.
Find the least whole number of days for the colony to exceed 400 cells.

$200 \times 1.05^n > 400$, so $1.05^n > 2$

$n > \frac{\ln 2}{\ln 1.05} = 14.21$

n must be a whole number, so n = 15

Answer: 15

7 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector $(52\mathbf{i} + 40\mathbf{j})$ km relative to a port O, and moves with constant velocity $(2\mathbf{i} - 2\mathbf{j})$ km h^{-1}. Ship B leaves O at t = 0 with constant speed 17 km h^{-1} and intercepts A.
Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: $(52\mathbf{i} + 40\mathbf{j}) + t(2\mathbf{i} - 2\mathbf{j})$. B at time t: t v_B (unknown velocity)
At interception B has travelled 17 t, so $|(52\mathbf{i} + 40\mathbf{j}) + t(2\mathbf{i} - 2\mathbf{j})| = 17$ t
Squaring and rearranging: $-281t^2 + 48t + 4304 = 0$, which gives t = 4 or a negative value
t = 4 (the other root is negative since B is faster than A)

A is then at $(52\mathbf{i} + 40\mathbf{j}) + 4(2\mathbf{i} - 2\mathbf{j}) = (60\mathbf{i} + 32\mathbf{j})$, so v_B $= \frac{1}{4}(60\mathbf{i} + 32\mathbf{j}) = (15\mathbf{i} + 8\mathbf{j})$

Answer: (a) 4 h (b) $(15\mathbf{i} + 8\mathbf{j})$ km h^{-1}

8 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{5}{(x-1)} < \frac{4}{(x-3)}$.

Multiply by the positive quantity $(x - 1)^2(x - 3)^2$:
$5(x - 1)(x - 3)^2 < 4(x - 3)(x - 1)^2$.
Move to one side and factorise: $(x - 1)(x - 3)(x - 11) < 0$.
Critical values: 1, 3, 11. Test the four regions.
Answer: $x < 1$ or $3 < x < 11$

9 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^{2x} \sin 3x$. Show that $dy/dx = e^{2x}(2 \sin 3x + 3 \cos 3x)$.
Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (2)e^{2x} \sin 3x + e^{2x} \times 3 \cos 3x$
$dy/dx = 0$ requires $2 \sin 3x + 3 \cos 3x = 0$

$\tan 3x = -\frac{3}{2}$, smallest positive 3x = 2.1588

x = 0.720
Second derivative test (or gradient change): a maximum
Answer: x = 0.720, maximum

Day 244 **A Level Maths: S Tier 10-a-day: Model Solutions**

10 Normal approximation to a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(200, 0.45).

(a) State why a normal approximation is suitable for X.

(b) Use a normal approximation to estimate $P(X \leq 82)$, to 3 significant figures.

np = 90 and nq = 110, both greater than 5 (n large, p not extreme)

Y ~ N(90, 49.5) with $\sigma = 7.036$

Apply the continuity correction:

$P(X \leq 82) \approx P(Y < 82.5) = P(Z < -1.066) = 0.143$

Answer: 0.143