Day 242 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 2x + 2$ and $g(x) = x^2 + 2$.
Solve $fg(x) = gf(x)$.

$fg(x) = 2(x^2 + 2) + 2 = 2x^2 + 6$.
$gf(x) = (2x + 2)^2 + 2 = 4x^2 + 8x + 6$.
Equate: $2x^2 + 8x = 0$.
Solve the quadratic: $x = -4$ or $x = 0$.
Answer: $x = -4$ or $x = 0$

2 Combined mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Class S has 20 students with mean mark 41 and standard deviation 3. Class R has 15 students with mean mark 78 and standard deviation 7.
(a) Find the mean mark of all 35 students.
(b) Find the standard deviation of all 35 students.

Σx for S $= 20 \times 41 = 820$, Σx for R $= 15 \times 78 = 1170$
Combined mean $= \frac{1990}{35} = 56.9$
$\Sigma x^2 = n(\sigma^2 + \text{mean}^2)$: $20(9 + 1681) + 15(49 + 6084) = 125795$
Variance $= \frac{125795}{35} - 56.857^2 = 361.41$, so $\sigma = 19.01$
Answer: (a) 56.9 (b) 19.0

3 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + 4x^2 + 11x + 8$.
Show that $(x + 1)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(-1) = -1 + 4 - 11 + 8 = 0$, so $(x + 1)$ is a factor.
$f(x) = (x + 1)(x^2 + 3x + 8)$.
For $x^2 + 3x + 8 = 0$: $b^2 - 4ac = 9 - 32 = -23 < 0$.
No real roots from the quadratic, so the only real root is $x = -1$.
Answer: Proved; the only real root is $x = -1$

4 Distance travelled with a change of direction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P moves on a straight line. At time t seconds its velocity is v m s^{-1}, where
$v = 2t^2 - 16t + 30$ for $0 \leq t \leq 6$
Find (a) the times at which P is at instantaneous rest, (b) the total distance travelled by P in the first 6 seconds.

$v = 2(t - 3)(t - 5)$, so $v = 0$ when $t = 3$ and $t = 5$
$v > 0$ for $0 < t < 3$, $v < 0$ for $3 < t < 5$, $v > 0$ for $t > 5$: direction changes at each
$s = \int v\,dt = \frac{2}{3}t^3 - 8t^2 + 30t$
Distances: 0 to 3: 36; 3 to 5: $\frac{8}{3}$; 5 to 6: $\frac{8}{3}$
Total $= 36 + \frac{8}{3} + \frac{8}{3}$
Answer: (a) $t = 3$, $t = 5$ (b) $\frac{124}{3}$ m

5 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is acute. $\cos A = \frac{5}{13}$ and $\sin B = \frac{21}{29}$.

Find the exact values of sin(A + B) and cos(A + B).

$\cos A = \frac{5}{13}$, $\sin A = \frac{12}{13}$; $\sin B = \frac{21}{29}$, $\cos B = \frac{20}{29}$

$\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{345}{377}$

$\cos(A + B) = \cos A \cos B - \sin A \sin B = -\frac{152}{377}$

Answer: $\sin(A + B) = \frac{345}{377}$, $\cos(A + B) = -\frac{152}{377}$

6 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $5y = 12x + k$ is a tangent to the circle

$(x - 2)^2 + (y - 4)^2 = 1$.

Find the two possible values of k.

Substitute $y = \frac{12x + k}{5}$: $(x - 2)^2 + (\frac{12x + k}{5} - 4)^2 = 1$.

Multiply by 25 and collect: a quadratic in x with a = 169.

Tangent ⇒ discriminant = 0. This simplifies to $(k + 4)^2 = 169$.

$k + 4 = \pm 13$.

Answer: $k = -17$ or $k = 9$

7 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $x^3 = 2x + 6$ has a root α.

(a) Show that the equation can be written as $x = \sqrt[3]{(2x + 6)}$.

(b) Use $x_{n+1} = \sqrt[3]{(2x_n + 6)}$ with $x_0 = 2$ to find x_1, x_2, x_3

to 4 d.p., and hence state α to 3 d.p.

Take the cube root of both sides to obtain $x = \sqrt[3]{(2x + 6)}$

$x_1 = 2.1544$, $x_2 = 2.1764$, $x_3 = 2.1795$

The iteration converges (successive values agree to 3 d.p.): $\alpha = 2.180$

Answer: $x_1 = 2.1544$, $x_2 = 2.1764$, $x_3 = 2.1795$; $\alpha = 2.180$

8 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two drones P and Q move so that, t seconds after launch, their position vectors in metres are

$\mathbf{r}_P = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + t\begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix} -2 \\ 10 \\ -5 \end{pmatrix} + t\begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} -4 \\ 9 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$

$-24 + t(6) = 0$, so $t = 4$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, with length $\sqrt{5}$

Answer: $t = 4$, minimum distance $= \sqrt{5}$ m

9 Rejected batches: binomial inside binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box holds 8 components, each faulty with probability 0.05, independently. A box is rejected if it has at least 2 faulty components.

(a) Find the probability that a box is rejected.

(b) 11 boxes are inspected. Find the probability that at least 3 are rejected.

Let $X \sim B(8, 0.05)$. $P(\text{rejected}) = P(X \geq 2) = 1 - P(X \leq 1) = 0.05724$

Let Y = number of rejected boxes, $Y \sim B(11, 0.05724)$

$P(Y \geq 3) = 1 - P(Y \leq 2) = 0.0219$

Answer: (a) 0.0572 (b) 0.0219

10 Non-uniform rod: centre of mass and tilting (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB of length 5 m and mass 6 kg rests on supports at C and D, where AC = 1 m and AD = 3 m, with $R_C = \frac{1}{3} R_D$.

Find (a) the distance of the centre of mass from A. A particle of mass m kg at B puts the rod on the point of tilting about D. (b) Find m.

$R_C = \frac{1}{3} R_D$. Moments about A: $R_C \times 1 + R_D \times 3 = 6\,g \times AG$

$R_D(\frac{1}{3} \times 1 + 3) = 6\,g \times AG$ with $R_D(1 + \frac{1}{3}) = 6\,g$, so $AG = \dfrac{\frac{1}{3} \times 1 + 3}{\frac{4}{3}} = 2.5$

Tilting about D: $R_C = 0$. Moments about D: $m\,g\,(5 - 3) = 6\,g\,(3 - 2.5)$

$m = \dfrac{6 \times 0.5}{2} = 1.5$

Answer: (a) 2.5 m (b) 1.5