Day 241 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Particle in equilibrium on two strings (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P of weight 49 N hangs in equilibrium from two light strings AP and BP attached to a horizontal ceiling at A and B (see diagram), where $\tan\alpha = \frac{5}{12}$ and $\tan\beta = \frac{3}{4}$.

Find (a) the tension in each string, (b) the greatest weight P could have if neither tension may exceed 90 N.

Horizontally: $T_1 \cos\alpha = T_2 \cos\beta$, so $T_1 \times \frac{12}{13} = T_2 \times \frac{4}{5}$

Vertically: $T_1 \sin\alpha + T_2 \sin\beta = 49$, and $T_2 = \frac{15}{13} T_1$

$T_1(\frac{5}{13} + \frac{15}{13} \times \frac{3}{5}) = 49$, so $T_1 = 45.5$ and $T_2 = 52.5$

The tension in BP is the larger: it is 52.5 when W = 49

Tensions are proportional to W, so $W_{max} = 49 \times \frac{90}{52.5} = 84$

Answer: (a) AP: 45.5 N, BP: 52.5 N (b) 84 N

2 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 10 pairs of values: $\Sigma x = 40$, $\Sigma y = 350$, $\Sigma x^2 = 305$, $\Sigma xy = 1835$.

(a) Find the equation of the regression line of y on x.

(b) The x values lie between 0 and 9. Estimate y when x = 10 and comment on its reliability.

$S_{xx} = 305 - \frac{40^2}{10} = 145$

$S_{xy} = 1835 - \frac{40 \times 350}{10} = 435$

$b = \frac{S_{xy}}{S_{xx}} = \frac{435}{145} = 3$

$a = \bar{y} - b\,\bar{x} = 35 - (3)(4) = 23$

$y = 23 + 3x$

At x = 10: y = 53. This is extrapolation (outside the data range), so may be unreliable

Answer: (a) y = 23 + 3x (b) 53, unreliable (extrapolation)

3 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 6 m and mass 40 kg rests on supports at C and D, where AC = 2 m and DB = 2 m. A girl of mass 50 kg stands on it at distance x m from A.

Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $40(1) = 50(x - 4)$, so $x = 4 + \frac{40 \times 1}{50} = 4.8$

Tilting about C (girl before C): $40(1) = 50(2 - x)$, so $x = 2 - \frac{40 \times 1}{50} = 1.2$

Answer: $1.2 \leq x \leq 4.8$

4 Replacing an observation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

20 recorded times (seconds) have mean 59 and standard deviation 6.

One value was recorded as 49 but should have been 57.

Find the correct mean and the correct standard deviation.

$\Sigma x = 20 \times 59 = 1180$, so corrected $\Sigma x = 1180 - 49 + 57 = 1188$

Correct mean $= \frac{1188}{20} = 59.40$

$\Sigma x^2 = 20(36 + 3481) = 70340$, corrected $\Sigma x^2 = 70340 - 2401 + 3249 = 71188$

Variance $= \frac{71188}{20} - 59.400^2 = 31.040$, $\sigma = 5.571$

Answer: mean 59.4, standard deviation 5.57

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5 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:
$x - y = 4$
$x^2 - xy - y^2 = -5$

From the first equation $x = 4 + y$.
Substitute into the second equation and collect terms:
this gives $-y^2 + 4y + 21 = 0$.
$y = -3$ or $y = 7$, giving $x = 1$ or $x = 11$.
Answer: (x, y) = (1, −3) and (11, 7)

6 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curves $y = e^x$ and $y = 16\,e^{-x}$ meet at the point P.
Find the exact area of the region bounded by the two curves and the y-axis.

$e^x = 16\,e^{-x}$ gives $e^{2x} = 16$, so $e^x = 4$, $x = \ln 4$

Area $= \int_0^{\ln 4} (16\,e^{-x} - e^x)\,dx = [-16\,e^{-x} - e^x]$

$= (-4 - 4) - (-16 - 1)$
Answer: 9

7 Proof by contradiction (logarithms) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\log_5 10$ is irrational.

Assume $\log_5 10 = \frac{m}{n}$, m, n positive integers.

Then $5^{\frac{m}{n}} = 10$, so $5^m = 10^n$.
Prime factors of 5^m and 10^n must be identical,
but 2 divides 10^n but 2 does not divide 5^m.
This is a contradiction, so $\log_5 10$ is irrational.
Answer: Proved

8 Points of inflection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the point of inflection of
$y = (x - 2)e^{-2x}$,
justifying that it is a point of inflection.

$dy/dx = e^{-2x}(1 - 2(x - 2))$
$d^2y/dx^2 = 2e^{-2x}(2(x - 2) - 2) = 0$, so $x = 3$
d^2y/dx^2 changes sign either side of $x = 3$
$y = e^{-6}$
Answer: $(3, e^{-6})$

9 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = 4x^2 - 32x + 62$
for $x \leq k$. Given that k is the greatest value for which
f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = 4(x - 4)^2 - 2$.
The vertex is at $x = 4$, so $k = 4$ (f is one-one on one side of it).

Let $y = f(x)$: $y + 2 = 4(x - 4)^2$, so $(x - 4) = \pm\sqrt{\frac{y + 2}{4}}$.

As $x \leq 4$, take the negative root: $x = 4 - \sqrt{\frac{y + 2}{4}}$.

Range of f = domain of f^{-1}: $f(4) = -2$, so $x \geq -2$.

Answer: $k = 4$; $f^{-1}(x) = 4 - \sqrt{\frac{x + 2}{4}}$; $x \geq -2$

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10 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 + xy + 2y^2 = 126$.
Find the coordinates of the points on C where the tangent is parallel to the y-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + y}{x + 4y}$

Vertical tangent when $x + 4y = 0$, so $x = -\frac{4y}{1}$

Substitute into the equation of C: $x^2 + (xy) + (2y^2) = 126$
Solve for the remaining coordinate
Answer: $(-12, 3)$ and $(12, -3)$