Day 240 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Transforming a graph: images of points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows y = f(x), made of three straight line segments. Find the coordinates of the images of A, B, C and D on the graph of

$y = -f(-\frac{1}{2}x - 2)$.

Inside f: replace x by $-\frac{1}{2}x - 2$, so x-coordinates become $(x + 2) \div (-\frac{1}{2})$.

Outside: y-coordinates become $-1 \times y$.

A: (8, −5) B: (−4, −3)

C: (−8, 4) D: (−16, −1)

Answer: A′(8, −5), B′(−4, −3), C′(−8, 4), D′(−16, −1)

2 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $x^3 = 4x + 5$ has a root α.

(a) Show that the equation can be written as $x = \sqrt[3]{(4x + 5)}$.

(b) Use $x_{n+1} = \sqrt[3]{(4x_n + 5)}$ with $x_0 = 3$ to find x_1, x_2, x_3 to 4 d.p., and hence state α to 3 d.p.

Take the cube root of both sides to obtain $x = \sqrt[3]{(4x + 5)}$

$x_1 = 2.5713$, $x_2 = 2.4817$, $x_3 = 2.4622$

The iteration converges (successive values agree to 3 d.p.): α = 2.457

Answer: $x_1 = 2.5713$, $x_2 = 2.4817$, $x_3 = 2.4622$; $\alpha = 2.457$

3 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (4 kg) lies on a rough table with μ = 0.25, and is joined by a light string over a smooth pulley (see diagram) to particle B (6 kg) on a smooth slope, α = 30°.

Released from rest, the string is cut when B has moved 3.25 m. Find the further distance A slides before stopping.

A: $T - 0.25 \times 4 \times 9.8 = 4a$. B: $6 \times 9.8 \times \frac{1}{2} - T = 6a$

Adding: $9.8(6 \times \frac{1}{2} - 0.25 \times 4) = 10a$, so $a = 1.96$

$T = 4(1.96 + 0.25 \times 9.8) = 17.64$

When cut: $v^2 = 2 \times 1.96 \times 3.25 = 12.74$. Then A decelerates at $\mu g = 2.45$

Further distance $= \frac{12.74}{2 \times 2.45} = 2.6$

Answer: 2.6 m

4 Identity then equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$.

(b) Hence solve $\tan\theta + \cot\theta = 5$ for $0 \le \theta < 360°$ (1 d.p.).

$\tan\theta + \cot\theta = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta} = \frac{2}{\sin 2\theta} = 2\operatorname{cosec} 2\theta$

$2\operatorname{cosec} 2\theta = 5$, so $\sin 2\theta = \frac{2}{5}$, $2\theta = 23.6°, 156.4°, 383.6°, 516.4°$

Halve each value

Answer: $\theta = 11.8°, 78.2°, 191.8°, 258.2°$

5 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (0.5t^2 - 6t)\mathbf{i} + (-3t^2 - 4t)\mathbf{j}$.

Find (a) the value of t at which P is moving parallel to $(\mathbf{i} + 2\mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (t - 6)\mathbf{i} + (-6t - 4)\mathbf{j}$

Parallel to $(\mathbf{i} + 2\mathbf{j})$: $(t - 6) \times 2 = (-6t - 4) \times 1$

$t(8) = 8$, so $t = 1$

At $t = 1$: $\mathbf{v} = (-5\mathbf{i} - 10\mathbf{j})$, speed $= \sqrt{(-5^2 + -10^2)}$

Answer: (a) t = 1 (b) 11.2 m s^{-1}

Day 240 **A Level Maths: S Tier 10-a-day: Model Solutions**

6 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{18}{2x^2 - 4x + 6}$ for all real x.

Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$2x^2 - 4x + 6 = 2(x - 1)^2 + 4$.

$(x - 1)^2 \geq 0$, so the denominator is at least 4, with equality at x = 1.

The denominator is always positive, so f(x) > 0.

Greatest f = $\frac{18}{4} = \frac{9}{2}$ at x = 1.

Answer: greatest value $\frac{9}{2}$ at x = 1; range $0 < f(x) \leq \frac{9}{2}$

7 Normal: find mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The times taken by a bus, in minutes are normally distributed. 25% are less than 49 and 5% are greater than 68.

Find the mean and standard deviation, each correct to 3 significant figures.

Let $X \sim N(\mu, \sigma^2)$. $P(X < 49) = 0.25$ gives $\frac{49 - \mu}{\sigma} = -0.6745$

$P(X > 68) = 0.05$ gives $\frac{68 - \mu}{\sigma} = 1.6449$

Subtract: $\frac{19}{\sigma} = 1.6449 - (-0.6745) = 2.3193$

$\sigma = 8.192$

$\mu = 49 - (-0.6745)(8.192) = 54.53$

Answer: $\mu = 54.5$, $\sigma = 8.19$

8 Tangent to a circle and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle $x^2 + y^2 - 6x - 2y - 15 = 0$

passes through P(7, −2). The tangent at P meets the axes at A and B. Find the area of triangle OAB.

Centre (3, 1). Gradient of radius CP = $-\frac{3}{4}$.

Tangent gradient = $\frac{4}{3}$: $4x - 3y = 34$.

x = 0: $y = -\frac{34}{3}$. y = 0: $x = \frac{17}{2}$.

Area = ½ × $\frac{17}{2}$ × $\frac{34}{3}$.

Answer: $\frac{289}{6}$

9 Trigonometric integrals with identities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use a suitable trigonometric identity to find the exact value of

$\int_0^{\pi/4} \tan^2 x \, dx$

Use $\tan^2 A \equiv \sec^2 A - 1$

Integrate term by term

Substitute the limits

Answer: $1 - \frac{1}{4}\pi$

10 Pdf of the form a + bx with a given mean (CIE)

The pdf of X is f(x) = a + bx for $0 \leq x \leq 2$, and 0 otherwise, where a and b are constants. E(X) = $\frac{7}{6}$.

(a) Find a and b.
(b) Find P(X < 1).

Total area = 1: 2 a + 2 b = 1

E(X) = $\int x(a + bx)\,dx$ = 2 a + $\frac{8}{3}$ b = $\frac{7}{6}$

Solving the simultaneous equations: a = $\frac{1}{4}$, b = $\frac{1}{4}$

P(X < 1) = $\int f(x)\,dx$ from 0 to 1 = $\frac{1}{4} \times 1 + \frac{1}{4} \times 0.5$

= $\frac{3}{8}$

Answer: (a) a = $\frac{1}{4}$, b = $\frac{1}{4}$ (b) $\frac{3}{8}$