Day 24 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Grouped median with an unknown frequency (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The table shows the times, t minutes, taken by some runners.
Using linear interpolation, the median is estimated as 60 minutes. Find the value of k.

Total = 40 + k, so the median class is 40 to 60

Median = $40 + \dfrac{\frac{(40+k)}{2} - (6+k)}{8} \times 20 = 60$

Simplify: $14 - k/2 = 8$

$k = 12$

Answer: k = 12

2 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2^{2x} - 12 \times 2^{x+1} + 128 = 0$,
giving each answer exactly and, where not an integer, to 3 s.f.

$2^{x+1} = 2 \times 2^x$. Let $u = 2^x$: $u^2 - 24u + 128 = 0$

$(u - 16)(u - 8) = 0$, so $2^x = 16$ or $2^x = 8$

Take logarithms: $x = \log_2 8 = 3$; $x = \log_2 16 = 4$

Answer: $x = \log_2 8 = 3$; $x = \log_2 16 = 4$

3 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $n^2 + 1$ is never a multiple of 3, for any integer n.

Every integer is 3q, 3q + 1, 3q + 2 for some integer q.

Squaring, $n^2 \equiv r^2 \pmod 3$, so:

$0 \to 0$; $1 \to 1$; $2 \to 1$ ($r \to n^2 \bmod 3$)

So $n^2 + 1 \equiv 1$ or $2 \pmod 3$.

No residue is 0, so $n^2 + 1$ is never a multiple of 3.

Answer: Proved

4 Speed decaying exponentially (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. The velocity of P is v m s^{-1} and its acceleration is $-0.2v^2$ m s^{-2}. At t = 0, v = 5.
Find (a) the time taken for the speed of P to halve, (b) the distance P travels in this time, to 3 s.f.

$\dfrac{dv}{dt} = -0.2v^2$: $\int \dfrac{dv}{v^2} = -0.2\int dt$, so $\dfrac{1}{v} = 0.2t + c$, and $\dfrac{1}{v} = 0.2t + \dfrac{1}{5}$

$v = 2.5$: $\dfrac{1}{v} = 0.4$, so $0.2t = 0.4 - 0.2$, giving $t = 1$

$s = \int \dfrac{1}{(0.2t + 1/5)}\, dt = \dfrac{1}{0.2} \ln(1 + 0.2 \times 5t)$

At t = 1: $s = \dfrac{1}{0.2} \ln 2$

Answer: (a) 1 s (b) 3.47 m

5 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2y + xy^2 = -12$.
Find the equation of the normal to C at the point (−3, −1),
in the form ax + by + c = 0 with integers a, b, c.

Differentiate implicitly with respect to x

At (−3, −1): $dy/dx = -\dfrac{7}{15}$

Normal gradient = $-1 \div (-\dfrac{7}{15}) = \dfrac{15}{7}$

$y - -1 = \dfrac{15}{7}(x - -3)$, then clear fractions

Answer: 15x − 7y + 38 = 0

6 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the term independent of x in the expansion of $(3x - \frac{1}{x^2})^{12}$.

General term: $C(12,r)(3x)^{12-r}(-\frac{1}{x^2})^r$.

Power of x: $1(12 - r) - 2r = 0$, so $r = 4$.

Term = $C(12,4) \times 3^8 \times (-1)^4 = 3247695$.

Answer: 3247695

7 Car on a hill: power and maximum speed (CIE)

A car of mass 800 kg, with engine power 10.6 kW, climbs a straight road with $\sin\alpha = \frac{1}{14}$ against a constant resistance of 500 N.

Find (a) the greatest steady speed up the hill, (b) the acceleration at 2.5 m s^{-1}. Take g = 9.8 m s^{-2}.

Steady speed: driving force = $500 + 800 \times 9.8 \times \frac{1}{14} = 1060$ N

$P = F v$, so $v = \frac{10600}{1060} = 10$

At 2.5 m s^{-1}: $F = \frac{10600}{2.5} = 4240$ N

$F - 1060 = 800 a$, so $a = \frac{3180}{800}$

Answer: (a) 10 m s^{-1} (b) 3.98 m s^{-2}

8 Circle touching both axes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle has its centre in the first quadrant and touches both the x-axis and the y-axis. It passes through the point (6, 3). Find the two possible equations of the circle.

Centre (r, r), radius r: $(x - r)^2 + (y - r)^2 = r^2$.

Substitute (6, 3): $(6 - r)^2 + (3 - r)^2 = r^2$.

$r^2 - 18r + 45 = 0$.

$(r - 3)(r - 15) = 0$, so $r = 3$ or $r = 15$.

Answer: $(x - 3)^2 + (y - 3)^2 = 9$ or $(x - 15)^2 + (y - 15)^2 = 225$

9 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 3 yellow counters and some black counters, n in total.

Two are taken without replacement. P(at least one yellow) = $\frac{8}{15}$. Find n.

P(neither yellow) = $\frac{(n-3)}{n} \times \frac{(n-4)}{(n-1)} = \frac{7}{15}$

$\frac{7}{15} n(n - 1) = (n - 3)(n - 4)$

Rearranging gives a quadratic in n whose roots are n = 10 and one other value

The other root is 2.25 so is rejected (not a valid number of counters)

n = 10

Answer: n = 10

10 Points of inflection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the point of inflection of

$y = (x - 3)e^{-2x}$,

justifying that it is a point of inflection.

$dy/dx = e^{-2x}(1 - 2(x - 3))$

$d^2y/dx^2 = 2e^{-2x}(2(x - 3) - 2) = 0$, so $x = 4$

d^2y/dx^2 changes sign either side of x = 4

$y = e^{-8}$

Answer: $(4, e^{-8})$