Day 239 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Range and greatest height to speed and angle (AQA, Edexcel, OCR, MEI, Eduqas)

A golf ball is hit from a point O on horizontal ground. It lands 48 m from O, and reaches a maximum height of 18 m above the ground.

Find the angle of projection and the speed of projection, to 3 s.f. Take g = 9.8 m s^{-2}.

Vertical: $0 = u_y^2 - 2 \times 9.8 \times 18$, so $u_y = 18.78$

Time of flight = $\frac{2u_y}{9.8} = 3.833$ s, so $u_x = \frac{48}{3.833} = 12.52$

$\tan\theta = \frac{u_y}{u_x} = 1.500$, so $\theta = 56.3°$

Speed = $\sqrt{(u_x^2 + u_y^2)} = 22.6$

Answer: θ = 56.3°, speed 22.6 m s^{-1}

2 Coded regression line (AQA, Edexcel, OCR, MEI)

Data pairs (x, y) are coded by $u = x - 7$ and $v = \frac{y}{5}$. The regression line of v on u is $v = 23 + 2u$.

(a) Find the regression line of y on x in the form $y = a + bx$.
(b) Use it to estimate y when x = 14.

Substitute $u = x - 7$ and $v = y/5$: $\frac{y}{5} = 23 + 2(x - 7)$

$y = 5[23 + 2x - 14] = 5 \times 9 + 5 \times 2x$
$y = 45 + 10x$
At x = 14: $y = 45 + 10 \times 14 = 185$
Answer: (a) y = 45 + 10x (b) 185

3 Projectile from a cliff (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected horizontally at 29.4 m s^{-1} from the top of a vertical cliff of height 78.4 m, and lands on horizontal ground.

Find (a) the time of flight and the distance from the cliff, (b) the speed and direction on landing. Take g = 9.8 m s^{-2}.

Vertically (up positive): $-78.4 = -4.9t^2$, so $t^2 = 16$
t = 4 s (the positive root)
Horizontal distance = 29.4 × 4 = 117.6 m
On landing: $v_x = 29.4$, $v_y = 0 - 9.8 \times 4 = -39.2$

speed = $\sqrt{(29.4^2 + 39.2^2)} = 49.0$, tan angle = $\frac{39.2}{29.4}$, angle = 53.1° below the horizontal

Answer: (a) 4 s, 117.6 m (b) 49.0 m s^{-1} at 53.1° below the horizontal

4 Polynomial inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x^4 - 8x^3 + 13x^2 + 30x - 72 < 0$.

Use the factor theorem to find roots: −2, 3, 4.
$f(x) = (x + 2)(x - 3)^2(x - 4)$.
Sign of f in each region (left to right): +, −, −, +.
Take the regions where f < 0 (include roots if ≤ or ≥).
Answer: $-2 < x < 3$ or $3 < x < 4$

5 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 15 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 90$, and after 4 minutes $\theta = 65$.
Find A and k, then find when $\theta = 61$ (3 s.f.).

$t = 0$: $15 + A = 90$ so $A = 75$

$t = 4$: $15 + 75e^{-4k} = 65$ so $e^{-4k} = \frac{2}{3}$

$k = \frac{1}{4} \ln \frac{3}{2} = 0.101$

$75\, e^{-kt} = 46$ so $t = \frac{1}{k} \ln \frac{75}{46}$

$t = 4.82$ minutes

Answer: A = 75, k = 0.101, t = 4.82 minutes

6 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx + 13}{x - 3}$ for $x \neq 3$. Given that f is self-inverse, find k, and hence solve $f(x) = x$.

Let $y = f(x)$: $y(x - 3) = kx + 13$, so $x(y - k) = 3y + 13$.

$f^{-1}(x) = \frac{3x + 13}{x - k}$.

$f = f^{-1}$ requires $k = 3$.

$f(x) = x$: $3x + 13 = x(x - 3)$, so $x^2 - 6x - 13 = 0$.

$x = 3 \pm \sqrt{22} = 3 \pm \sqrt{22}$.

Answer: k = 3; x = $3 \pm \sqrt{22}$

7 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (1, 4, −3), (3, 1, −3), (−3, −1, −2).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$, $BC = \begin{pmatrix} -6 \\ -2 \\ 1 \end{pmatrix}$; $BA \cdot BC = 6$

$|BA| = \sqrt{13}$, $|BC| = \sqrt{41}$

$\cos ABC = \frac{6}{\sqrt{13}\sqrt{41}} = 0.2599$, so $ABC = 74.9°$

$|BA \times BC| = \left|\begin{pmatrix} 3 \\ 2 \\ 22 \end{pmatrix}\right| = \sqrt{497}$

Area $= \frac{1}{2}|BA \times BC|$

Answer: ABC = 74.9°, area = $\frac{1}{2}\sqrt{497}$

8 Normal: find mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The times taken by a bus, in minutes are normally distributed. 5% are less than 26 and 10% are greater than 62.
Find the mean and standard deviation, each correct to 3 significant figures.

Let $X \sim N(\mu, \sigma^2)$. $P(X < 26) = 0.05$ gives $\frac{26 - \mu}{\sigma} = -1.6449$

$P(X > 62) = 0.10$ gives $\frac{62 - \mu}{\sigma} = 1.2816$

Subtract: $\frac{36}{\sigma} = 1.2816 - (-1.6449) = 2.9264$

$\sigma = 12.30$

$\mu = 26 - (-1.6449)(12.30) = 46.23$

Answer: μ = 46.2, σ = 12.3

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9 Remainder theorem (quadratic divisor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

When a polynomial f(x) is divided by (x + 1)
the remainder is 7; when divided by (x − 2)
the remainder is −8. Find the remainder when
f(x) is divided by (x + 1)(x − 2).

f(x) = (x + 1)(x − 2)Q(x) + px + q.
f(−1) = 7: −p + q = 7.
f(2) = −8: 2p + q = −8.
Subtracting: −3p = 15, so p = −5, q = 2.
Answer: −5x + 2

10 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^{2x} \sin x$. Show that $dy/dx = e^{2x}(2 \sin x + \cos x)$.
Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (2)e^{2x} \sin x + e^{2x} \times \cos x$
$dy/dx = 0$ requires $2 \sin x + 1 \cos x = 0$

$\tan x = -\frac{1}{2}$, smallest positive x = 2.6779

x = 2.68
Second derivative test (or gradient change): a maximum
Answer: x = 2.68, maximum