Day 237 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car P passes a point O travelling at a constant speed of 20 m s^{-1}. 10 s later, car Q leaves O from rest with constant acceleration 3 m s^{-2}, travelling in the same direction.
Q overtakes P at the point X. Find (a) the time Q takes to reach X, (b) the distance OX, (c) the speed of Q at X.

Let Q travel for t s. P: distance = 20(t + 10); Q: distance = ½ × 3 × t^2
Equate: $3t^2 - 40t - 400 = 0$, so $(t - 20)(3t + 20) = 0$
t = 20 (the other root is negative)
OX = 3 × 20^2 / 2 = 600 m, and speed = 3 × 20 = 60
Answer: (a) 20 s (b) 600 m (c) 60 m s^{-1}

2 Double angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos 2\theta - 5\sin\theta - 3 = 0$
for $0 \leq \theta < 360°$.

Use $\cos 2\theta \equiv 1 - 2\sin^2\theta$: $2\sin^2\theta + 5\sin\theta + 2 = 0$
$(2\sin\theta + 1)(\sin\theta + 2) = 0$
$\sin\theta = -\frac{1}{2}$ or $\sin\theta = -2$ (impossible, $|\sin\theta| \leq 1$)
Using the CAST diagram for $0 \leq \theta < 360°$
Answer: θ = 210°, 330°

3 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (2 kg) lies on a rough table with $\mu = 0.25$, and is joined by a light string over a smooth pulley (see diagram) to particle B (3 kg) on a smooth slope, $\alpha = 30°$.
Released from rest, the string is cut when B has moved 3.5 m. Find the further distance A slides before stopping.

A: $T - 0.25 \times 2 \times 9.8 = 2a$. B: $3 \times 9.8 \times \frac{1}{2} - T = 3a$
Adding: $9.8(3 \times \frac{1}{2} - 0.25 \times 2) = 5a$, so $a = 1.96$
$T = 2(1.96 + 0.25 \times 9.8) = 8.82$
When cut: $v^2 = 2 \times 1.96 \times 3.5 = 13.72$. Then A decelerates at $\mu g = 2.45$
Further distance = $\frac{13.72}{2 \times 2.45} = 2.8$
Answer: 2.8 m

4 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is acute. $\sin A = \frac{15}{17}$ and $\sin B = \frac{4}{5}$.
Find the exact values of cos(A + B) and sin(A + B).

$\cos A = \frac{8}{17}$, $\sin A = \frac{15}{17}$; $\sin B = \frac{4}{5}$, $\cos B = \frac{3}{5}$
$\cos(A + B) = \cos A \cos B - \sin A \sin B = -\frac{36}{85}$
$\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{77}{85}$
Answer: $\cos(A + B) = -\frac{36}{85}$, $\sin(A + B) = \frac{77}{85}$

5 Partial fractions (improper) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{3x^3 - 15x^2 + 21x - 15}{(x - 1)(x - 3)}$ in the form $ax + b + \frac{C}{(x - 1)} + \frac{D}{(x - 3)}$.

Divide: the numerator has degree 3, the denominator degree 2.
$(3x^3 - 15x^2 + 21x - 15) \div (x^2 - 4x + 3)$ gives quotient $3x - 3$.
Remainder: -6.
Put x = 1: remainder = $C(-2)$ so C = 3. Put x = 3: D = −3.
Answer: $3x - 3 + \frac{3}{(x - 1)} - \frac{3}{(x - 3)}$

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6 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 3 green counters and some grey counters, n in total.

Two are taken without replacement. P(at least one green) = $\frac{5}{11}$. Find n.

P(neither green) = $\frac{(n-3)}{n} \times \frac{(n-4)}{(n-1)} = \frac{6}{11}$

$\frac{6}{11}\, n(n-1) = (n-3)(n-4)$

Rearranging gives a quadratic in n whose roots are n = 12 and one other value

The other root is 2.2 so is rejected (not a valid number of counters)

n = 12

Answer: n = 12

7 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + ax^2 + bx - 8$, where a and b are constants.

$(x - 4)$ is a factor of f(x), and f(x) leaves remainder 0 when divided by $(x - 2)$.

Find a and b, and hence solve f(x) = 0.

f(4) = 0: $64 + 16a + 4b - 8 = 0$, so $16a + 4b + 56 = 0$.

f(2) = 0: $4a + 2b = 0$.

Solving: $a = -7$, $b = 14$.

$f(x) = x^3 - 7x^2 + 14x - 8 = (x - 1)(x - 2)(x - 4)$

Answer: a = −7, b = 14; x = 1, 2, 4

8 Indices: simultaneous equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously, giving integer x and y:

$8^x \times 2^y = 512$

$9^x \div 3^y = 3$

Write each side with prime bases: $512 = 2^9$, $3 = 3^1$.

First: $2^{3x} \times 2^{1y} = 2^9$, so $3x + 1y = 9$.

Second: $3^{2x - 1y} = 3^1$, so $2x - 1y = 1$.

Solving the pair of linear equations: x = 2, y = 3.

Answer: x = 2, y = 3

9 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each bulb is faulty with probability 0.1, independently.

Find the least number n of bulbs that must be tested so that the probability of finding at least one faulty is greater than 0.99.

P(at least one faulty) = $1 - 0.9^n > 0.99$

$0.9^n < 0.01$

$n \ln 0.9 < \ln 0.01$

Dividing by ln 0.9 (negative) reverses the inequality: n > 43.71

n = 44

Answer: n = 44

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10 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two drones P and Q move so that, t seconds after launch, their position vectors in metres are

$\mathbf{r}_P = \begin{pmatrix} 0 \\ -5 \\ -1 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} = 0$

$-22 + t(22) = 0$, so $t = 1$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$, with length $\sqrt{2}$

Answer: t = 1, minimum distance = $\sqrt{2}$ m