Day 235 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Double angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos 2\theta - 5\cos\theta + 3 = 0$
for $0 \leq \theta < 360°$.

Use $\cos 2\theta \equiv 2\cos^2\theta - 1$: $2\cos^2\theta - 5\cos\theta + 2 = 0$
$(2\cos\theta - 1)(\cos\theta - 2) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = 2$ (impossible, $|\cos\theta| \leq 1$)

Using the CAST diagram for $0 \leq \theta < 360°$
Answer: $\theta = 60°, 300°$

2 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass M kg, carrying a passenger of mass 80 kg, hangs from a vertical cable. The tension in the cable is 5280 N and the floor exerts a force of 880 N on the passenger.
Find (a) the acceleration of the lift, with its direction, (b) the value of M. Take g = 9.8 m s^{-2}.

Passenger (upwards positive): $880 - 80 \times 9.8 = 80a$, so $a = 1.2$
$a = 1.2$ m s^{-2}, so the acceleration is upwards
Lift and passenger: $5280 - (M + 80) \times 9.8 = (M + 80) \times 1.2$

$(M + 80)(9.8 + 1.2) = 5280$, so $M + 80 = \frac{5280}{11} = 480$

Answer: (a) 1.2 m s^{-2} upwards (b) M = 400

3 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes
(dashed) and a point P. C has equation

$y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes $x = -1$ and $y = -1$, so $p = 1$ and $q = -1$.

$y = \frac{a}{x + 1} - 1$.

P$(-2, -4)$: $-4 = \frac{a}{-1} - 1$, so $a = 3$.

Answer: a = 3, p = 1, q = −1

4 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A cone with its vertex downwards and semi-vertical angle α, where

$\tan\alpha = \frac{1}{2}$, is filled at 15π cm^3s^{-1}. Find the rate at which

the depth h is rising when h = 5 cm. ($V = \frac{1}{3}\pi r^2 h$)

$r = \frac{h}{2}$, so $V = \frac{1}{3}\pi \times \frac{h^2}{4} \times h = \frac{\pi}{12}h^3$

$dV/dt = \frac{\pi}{4}h^2\, dh/dt$, so $15\pi = \frac{\pi}{4} \times 25 \times dh/dt$

$dh/dt = \frac{60}{25}$

Answer: $\frac{12}{5}$ cm s^{-1}

5 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both

$x^2 + 6x + 5 < 0$ and $-x + 3 \leq 7$.

Quadratic: $(x + 5)(x + 1) < 0$ gives $-5 < x < -1$.
Linear: $-x + 3 \leq 7$ gives $x \geq -4$. (the inequality reverses as we divide by a negative)
Both must hold: take the overlap of the two sets.
Answer: $-4 \leq x < -1$

6 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(30, \frac{1}{3})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{30 - r}{r + 1} \times \frac{1}{2}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(30, r + 1)}{C(30, r)} \times \frac{p}{q} = \frac{30 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{1}{3}$ and $q = \frac{2}{3}$, $p \div q = \frac{1}{2}$

The ratio exceeds 1 when $(30 - r)\frac{1}{3} > (r + 1)\frac{2}{3}$, that is $r < \frac{28}{3}$

So probabilities rise up to r = 10 and then fall; the mode is 10

P(X = 10) = 0.153

Answer: 10 with probability 0.153

7 Equations in disguise (substitution) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x - 9\sqrt{x} + 18 = 0$.

Let $u = \sqrt{x}$: $u^2 - 9u + 18 = 0$.

$(u - 3)(u - 6) = 0$, so $\sqrt{x} = 3$ or $\sqrt{x} = 6$.

Square: x = 9 or x = 36.

Answer: x = 9 or x = 36

8 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ax^n$. The line of best fit for lg y against lg x is lg y = 0.65 + 2 lg x.

(a) Find a and n, giving a to 3 significant figures.

(b) Estimate y when x = 20, to 3 significant figures.

$y = ax^n$ gives lg y = lg a + n lg x, so the gradient is n and the intercept is lg a

n = 2

lg a = 0.65 so $a = 10^{0.65} = 4.467$

x = 20: $y = 4.467 \times 20^2$

= 1790

Answer: (a) a = 4.47, n = 2 (b) 1790

9 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two boats P and Q move so that, t hours after noon, their position vectors in km are

$\mathbf{r}_P = \begin{pmatrix}1\\5\end{pmatrix} + t\begin{pmatrix}-2\\1\end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix}6\\10\end{pmatrix} + t\begin{pmatrix}-3\\-2\end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}5\\5\end{pmatrix} + t\begin{pmatrix}-1\\-3\end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix}-1\\-3\end{pmatrix} = 0$

$-20 + t(10) = 0$, so t = 2

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}3\\-1\end{pmatrix}$, with length $\sqrt{10}$

Answer: t = 2, minimum distance = $\sqrt{10}$ km

10 Non-uniform rod: centre of mass and tilting (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB of length 5 m and mass 5 kg rests on supports at C and D, where AC = 2 m and AD = 4 m, with $R_C = 3R_D$.

Find (a) the distance of the centre of mass from A. A particle of mass m kg at B puts the rod on the point of tilting about D. (b) Find m.

$R_C = 3R_D$. Moments about A: $R_C \times 2 + R_D \times 4 = 5g \times AG$

$R_D(3 \times 2 + 4) = 5g \times AG$ with $R_D(1 + 3) = 5g$, so $AG = \dfrac{3 \times 2 + 4}{4} = 2.5$

Tilting about D: $R_C = 0$. Moments about D: $mg(5 - 4) = 5g(4 - 2.5)$

$m = \dfrac{5 \times 1.5}{1} = 7.5$

Answer: (a) 2.5 m (b) 7.5