Day 234 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Continuous random variable: pdf with unknown constant (CIE)

The pdf of X is $f(x) = kx^2$ for $0 \le x \le 2$, and 0 otherwise.
(a) Find k.
(b) Find E(X) and the standard deviation of X.
(c) Find the median, to 3 significant figures.

(a) $\int kx^2\,dx$ from 0 to 2 $= k \times \frac{2^3}{3} = 1$, so $k = \frac{3}{8}$

(b) $E(X) = \int \frac{3}{8}x^3\,dx = \frac{3}{2}$

$E(X^2) = \frac{12}{5}$, $\text{Var}(X) = \frac{12}{5} - \left(\frac{3}{2}\right)^2 = \frac{3}{20}$, $\sigma = 0.3873$

(c) Median m: $\int f$ from 0 to m $= \frac{m^3}{8} = \frac{1}{2}$

$m = 2 \times (1/2)^{1/3} = 1.587$

Answer: (a) $k = \frac{3}{8}$ (b) $E(X) = \frac{3}{2}$, $\sigma = 0.387$ (c) 1.59

2 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A cone with its vertex downwards and semi-vertical angle α, where $\tan\alpha = \frac{1}{2}$, is filled at 15π cm^{3}s^{-1}. Find the rate at which the depth h is rising when h = 3 cm. ($V = \frac{1}{3}\pi r^2 h$)

$r = \frac{h}{2}$, so $V = \frac{1}{3}\pi \times \frac{h^2}{4} \times h = \frac{\pi}{12}h^3$

$dV/dt = \frac{\pi}{4}h^2\,dh/dt$, so $15\pi = \frac{\pi}{4} \times 9 \times dh/dt$

$dh/dt = \frac{60}{9}$

Answer: $\frac{20}{3}$ cm s^{-1}

3 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 450 kg, carrying a passenger of mass 90 kg, is supported by a vertical cable. The lift is moving downwards and accelerating at 1.4 m s^{-2}.
Find (a) the force the floor exerts on the passenger, (b) the tension in the cable. Take g = 9.8 m s^{-2}.

Take the direction of the acceleration as positive: it is downwards
Passenger: $90 \times 9.8 - R = 90 \times 1.4$
$R = 90 \times (9.8 - 1.4) = 756$
Lift and passenger: $540 \times 9.8 - T = (450 + 90) \times 1.4$
$T = 540 \times 8.4 = 4536$
Answer: (a) 756 N (b) 4536 N

4 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 24 and upper quartile 39. An outlier is a value more than 1.5 × IQR beyond the nearest quartile.
(a) Find the range of values that are not outliers.
(b) 8 observations have mean 29. The largest is 67, an outlier, and it is removed. Find the new mean.

IQR = 39 − 24 = 15, so 1.5 × IQR = 22.5
Lower limit = 24 − 22.5 = 1.5, upper limit = 39 + 22.5 = 61.5
$\Sigma x = 8 \times 29 = 232$, so without 67: $\Sigma x = 165$
New mean $= \frac{165}{7}$

Answer: (a) 1.5 to 61.5 (b) 23.57

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5 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 8 m and mass 30 kg rests on supports at C and D, where AC = 2 m and DB = 3 m. A girl of mass 50 kg stands on it at distance x m from A.

Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $30(1) = 50(x - 5)$, so $x = 5 + \frac{30 \times 1}{50} = 5.6$

Tilting about C (girl before C): $30(2) = 50(2 - x)$, so $x = 2 - \frac{30 \times 2}{50} = 0.8$

Answer: $0.8 \leq x \leq 5.6$

6 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of

$y = 2x^2 \ln x$ for $x > 0$,

and determine its nature.

$dy/dx = 2x^1(2 \ln x + 1) = 0$ gives $\ln x = -\frac{1}{2}$

$y = -e^{-1}$ when $x = \frac{1}{\sqrt{e}}$

Second derivative at this point is positive, so it is a minimum

Answer: $(\frac{1}{\sqrt{e}}, -e^{-1})$, minimum

7 Transformations of y = x^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Describe a sequence of two transformations that maps the curve $y = x^2$ onto the curve

$y = -2x^2 - 20x - 46$.

Complete the square: $y = -2(x + 5)^2 + 4$.

Step 1: $y = x^2 \rightarrow y = -2x^2$: a stretch, scale factor 2, parallel to the y-axis and a reflection in the x-axis.

Step 2: replace x by (x + 5) and add 4: translation by $\begin{pmatrix} -5 \\ 4 \end{pmatrix}$.

Answer: a stretch, scale factor 2, parallel to the y-axis and a reflection in the x-axis, then translation $\begin{pmatrix} -5 \\ 4 \end{pmatrix}$

8 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 3 and common ratio $\frac{5}{4}$.

Find the least n for which $S_n > 16704$.

$S_n = \frac{3(\frac{5}{4}^n - 1)}{\frac{1}{4}} > 16704.$

$\frac{5}{4}^n > 1393.$

$n > \frac{\log(1393)}{\log(1.25)} = 32.442.$

So the least n is 33.

Answer: n = 33

9 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 + 3xy + 4y^2 = 112$.
Find the coordinates of the points on C where the tangent is parallel to the y-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + 3y}{3x + 8y}$

Vertical tangent when $3x + 8y = 0$, so $x = -\frac{8y}{3}$

Substitute into the equation of C: $x^2 + (3xy) + (4y^2) = 112$
Solve for the remaining coordinate
Answer: $(-16, 6)$ and $(16, -6)$

10 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation

$y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes $x = -1$ and $y = -2$, so $p = 1$ and $q = -2$.

$y = \frac{a}{x + 1} - 2$.

P(1, −1): $-1 = \frac{a}{2} - 2$, so $a = 2$.

Answer: $a = 2$, $p = 1$, $q = -2$