Day 233 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each bulb is faulty with probability 0.08, independently.
Find the least number n of bulbs that must be tested so that the probability of finding at least two faulty is greater than 0.95.

Let $X \sim B(n, 0.08)$. $P(X \geq 2) = 1 - 0.92^n - n(0.08)(0.92)^{n-1}$
This has no simple algebraic solution, so test values of n
n = 57: $P(X \geq 2) = 0.9486$, which is not greater than 0.95
n = 58: $P(X \geq 2) = 0.9520$, which is greater than 0.95
$P(X \geq 2)$ increases with n, so the least value is 58
Answer: n = 58

2 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + ax^2 + bx + 36$, where a and b are constants.
$(x + 3)$ is a factor of f(x), and f(x) leaves remainder 40 when divided by $(x + 1)$.
Find a and b, and hence solve $f(x) = 0$.

$f(-3) = 0$: $-27 + 9a - 3b + 36 = 0$, so $9a - 3b + 9 = 0$.
$f(-1) = 40$: $a - b + 35 = 40$.
Solving: $a = -4$, $b = -9$.
$f(x) = x^3 - 4x^2 - 9x + 36 = (x + 3)(x - 3)(x - 4)$
Answer: a = −4, b = −9; x = −3, 3, 4

3 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level with speed 78.4 m s^{-1}.
Find the length of time for which the ball is more than 269.5 m above the ground, and the greatest height it reaches. Take g = 9.8 m s^{-2}.

At height 269.5 m: $v^2 = 78.4^2 - 2 \times 9.8 \times 269.5 = 864.36$, so $v = 29.4$
Time from this height to the top: $t = \frac{29.4}{9.8} = 3$ s
By symmetry the time above this height is $2 \times 3 = 6$ s
Greatest height = $\frac{78.4^2}{(2 \times 9.8)} = 313.6$
Answer: 6 s, 313.6 m

4 Cubic graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the cubic y = f(x), which touches the x-axis once. Find f(x), giving your answer in expanded form.

Crosses at $x = -4$ and touches at $x = -3$, so $f(x) = k(x + 4)(x + 3)^2$.
At (0, −36): $-36 = k(4)(3)^2 = 36k$, so $k = -1$.
$f(x) = -(x + 4)(x + 3)^2$, then expand.
Answer: $f(x) = -x^3 - 10x^2 - 33x - 36$

5 Triple angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (triple angle))

Given that $\sin 3\theta \equiv 3 \sin \theta - 4 \sin^3\theta$, solve
$\sin 3\theta = -\sin \theta$ for $0 < \theta < \pi$, in terms of π.

Substitute: $\sin \theta(3 - 4\sin^2\theta + 1) = 0$
$\sin \theta = 0$ or $\sin^2\theta = 1$
$\sin \theta = 0$ gives no solutions in the range
$\sin^2\theta = 1$ gives $\frac{\pi}{2}$
Answer: $\theta = \frac{\pi}{2}$

6 Slide down a rough slope then along rough ground (CIE)

A particle starts from rest at the top of a rough slope of length 25 m ($\sin\alpha = \frac{4}{5}$, $\mu = 0.25$), then slides along rough horizontal ground with the same μ.

Use energy to find (a) its speed at the foot of the slope (3 s.f.), (b) the distance it slides on the ground. Take $g = 9.8$ m s^{-2}.

On the slope: $\frac{1}{2} m v^2 = m g \times 25 \sin\alpha - \mu m g \cos\alpha \times 25$

$v^2 = 2 \times 9.8 \times 25(\frac{4}{5} - 0.25 \times \frac{3}{5}) = 318.5$, so $v = 17.85$

On the ground: $\frac{1}{2} m v^2 = \mu m g x$, so $x = \frac{v^2}{2\mu g} = \frac{318.5}{4.9} = 65$

Answer: (a) 17.8 m s^{-1} (b) 65 m

7 Cumulative distribution function (CIE)

The cdf of X is $F(x) = 0$ for $x < 0$, $F(x) = \frac{x^3}{64}$ for $0 \le x \le 4$, and $F(x) = 1$ for $x > 4$.

(a) Find the pdf of X.
(b) Find E(X).
(c) Find $P(1 < X < 2)$ and the median, to 3 significant figures.

(a) $f(x) = F'(x) = \frac{3}{64} x^2$ for $0 \le x \le 4$

(b) $E(X) = \int x f(x)\,dx$ from 0 to 4 $= 3$

(c) $P = F(2) - F(1) = 0.1094$

Median m: $F(m) = 1/2$, so $m = 4 \times (1/2)^{1/3}$

$m = 3.175$

Answer: (a) $f(x) = \frac{3}{64} x^2$ (b) 3 (c) 0.109, median 3.17

8 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (0, 0, −3), (4, 3, −2), (0, −3, 1).

Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix} -4 \\ -3 \\ -1 \end{pmatrix}$, $BC = \begin{pmatrix} -4 \\ -6 \\ 3 \end{pmatrix}$; $BA \cdot BC = 31$

$|BA| = \sqrt{26}$, $|BC| = \sqrt{61}$

$\cos ABC = \frac{31}{\sqrt{26}\sqrt{61}} = 0.7784$, so $ABC = 38.9°$

$|BA \times BC| = \left|\begin{pmatrix} -15 \\ 16 \\ 12 \end{pmatrix}\right| = \sqrt{625}$

Area $= \frac{1}{2}|BA \times BC|$

Answer: ABC = 38.9°, area = $\frac{25}{2}$

9 Indices: quadratic in a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2^{2x} - 6 \times 2^x + 8 = 0$.

Let $y = 2^x$. Then $y^2 - 6y + 8 = 0$.

$(y - 2)(y - 4) = 0$, so $y = 2$ or $y = 4$.

$2^x = 2$ gives $x = 1$; $2^x = 4$ gives $x = 2$.

Answer: x = 1 or x = 2

10 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two drones P and Q move so that, t seconds after launch, their position vectors in metres are

$\mathbf{r}_P = \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix} 0 \\ -7 \\ 5 \end{pmatrix} + t\begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0$

$-5 + t(1) = 0$, so $t = 5$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$, with length $\sqrt{4}$

Answer: t = 5, minimum distance = 2 m