Day 231 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Arithmetic terms forming a geometric sequence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first term of an arithmetic sequence is 4 and its common difference d is non-zero. The 1st, 5th and 13th terms are the first three terms of a geometric sequence. Find d and the common ratio.

Terms: 4, 4 + 4d, 4 + 12d.

Geometric: $(4 + 4d)^2 = (4)(4 + 12d)$.

Expand: $16d^2 + 32d + 16 = 48d + 16$, so $d(16d - 16) = 0$.

$d \neq 0$, so $d = 1$. Ratio = $\frac{8}{4} = 2$.

Answer: d = 1, ratio = 2

2 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each bulb is faulty with probability 0.12, independently.
Find the least number n of bulbs that must be tested so that the probability of finding at least two faulty is greater than 0.9.

Let $X \sim B(n, 0.12)$. $P(X \geq 2) = 1 - 0.88^n - n(0.12)(0.88)^{n-1}$
This has no simple algebraic solution, so test values of n
n = 30: $P(X \geq 2) = 0.8900$, which is not greater than 0.9
n = 31: $P(X \geq 2) = 0.9006$, which is greater than 0.9
$P(X \geq 2)$ increases with n, so the least value is 31
Answer: n = 31

3 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 6 strips to estimate

$\int_0^1 \frac{1}{\sqrt{1+x}}\,dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

h = $\frac{1}{6}$; ordinates at x = 0, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$, …, 1

Sum = $\frac{h}{2}[y_0 + y_6 + 2(y_1 + \ldots + y_5)] = 0.8292$

f''(x) > 0 on the interval, so the curve is convex and the trapezia lie above the curve
Answer: 0.8292; overestimate

4 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ab^x$. The line of best fit for lg y against x is lg y = 0.65 + 0.4x.
(a) Find a and b to 3 significant figures.
(b) Estimate y when x = 14, to 3 significant figures.
(c) Interpret the value of b.

$y = ab^x$ gives $\lg y = \lg a + x \lg b$
$\lg a = 0.65$ so $a = 10^{0.65} = 4.467$
$\lg b = 0.4$ so $b = 10^{0.4} = 2.512$
x = 14: $\lg y = 0.65 + 0.4 \times 14 = 6.25$
$y = 10^{6.25} = 1780000$
b is the factor y is multiplied by each time x increases by 1 (growth of 150%)
Answer: (a) a = 4.47, b = 2.51 (b) 1780000 (c) y grows by a factor of 2.51 per unit of x

5 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates $(-3, 3, -3)$, $(-3, 2, 0)$, $(1, -3, 0)$.
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$\text{BA} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$, $\text{BC} = \begin{pmatrix} 4 \\ -5 \\ 0 \end{pmatrix}$; $\text{BA} \cdot \text{BC} = -5$

$|\text{BA}| = \sqrt{10}$, $|\text{BC}| = \sqrt{41}$

$\cos \text{ABC} = \dfrac{-5}{\sqrt{10}\sqrt{41}} = -0.2469$, so $\text{ABC} = 104.3°$

$|\text{BA} \times \text{BC}| = \left|\begin{pmatrix} -15 \\ -12 \\ -4 \end{pmatrix}\right| = \sqrt{385}$

$\text{Area} = \frac{1}{2}|\text{BA} \times \text{BC}|$

Answer: ABC = 104.3°, area = $\frac{1}{2}\sqrt{385}$

6 Geometric series: sum to infinity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has sum to infinity 18 and second term 4. Find the possible values of the common ratio r and the first term a in each case.

$a = \frac{4}{r}$. $S_\infty = \dfrac{a}{1-r} = \dfrac{4}{r(1-r)} = 18$.

$18r^2 - 18r + 4 = 0$.

$(3r - 1)(3r - 2) = 0$: $r = \frac{1}{3}$ or $r = \frac{2}{3}$.

Need $|r| < 1$: $\frac{1}{3}$, $\frac{2}{3}$ valid.

Answer: $r = \frac{1}{3}$, $a = 12$ or $r = \frac{2}{3}$, $a = 6$

7 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 2t^2 - 2t$, $y = 2t^3 - 2t$.
Find the equation of the tangent at the point where $t = -1$,
in the form $y = mx + c$.

$dx/dt = 4t - 2$, $dy/dt = 6t^2 - 2$

At $t = -1$: $dy/dx = \frac{4}{-6} = -\frac{2}{3}$; point $(4, 0)$

$y - 0 = (-\frac{2}{3})(x - 4)$

Answer: $y = -\frac{2}{3}x + \frac{8}{3}$

8 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time t = 0 hours, A is at $(-4\mathbf{i} + 6\mathbf{j})$ km relative to a fixed origin and B is at $(-17\mathbf{i} - 10\mathbf{j})$ km. A has velocity $(-\mathbf{i} + 2\mathbf{j})$ km h^{-1} and B has velocity $(3\mathbf{i} + 5\mathbf{j})$ km h^{-1}.
Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (-13\mathbf{i} - 16\mathbf{j}) + t(4\mathbf{i} + 3\mathbf{j})$

$|\mathbf{d}|^2 = (4t - 13)^2 + (3t - 16)^2 = 25t^2 - 200t + 425$

Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\dfrac{-100}{25} = 4$

$\mathbf{d} = (-13\mathbf{i} - 16\mathbf{j}) + 4(4\mathbf{i} + 3\mathbf{j}) = (3\mathbf{i} - 4\mathbf{j})$, so least distance = 5

Answer: (a) t = 4 h (b) 5 km

9 Proof by contradiction (logarithms) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\log_8 15$ is irrational.

Assume $\log_8 15 = \frac{m}{n}$, m, n positive integers.

Then $8^{\frac{m}{n}} = 15$, so $8^m = 15^n$.

Prime factors of 8^m and 15^n must be identical,

but 3 divides 15^n but 3 does not divide 8^m.

This is a contradiction, so $\log_8 15$ is irrational.

Answer: Proved

10 Projectile through a given point: two angles (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 29 m s^{-1}, at angle θ above the horizontal. It passes through the point P, 58 m horizontally and 18 m vertically from O.

Find the two possible values of $\tan\theta$, and the two possible values of θ to 1 d.p. Take g = 9.8 m s^{-2}.

Horizontal: $x = 29\cos\theta \times t$. Vertical: $y = 29\sin\theta \times t - 4.9t^2$

Eliminating t: $y = x\tan\theta - \frac{9.8x^2}{2u^2}(1 + \tan^2\theta)$

$\frac{9.8 \times 58^2}{2 \times 29^2} = 19.6$, so $18 = 58\tan\theta - 19.6(1 + \tan^2\theta)$

$19.6\tan^2\theta - 58\tan\theta + 37.6 = 0$

$\tan\theta = \frac{47}{49}$ or 2, so $\theta = 43.8°$ or $63.4°$

Answer: $\tan\theta = \frac{47}{49}$ or 2; $\theta = 43.8°$ or $63.4°$