Day 23 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Particle on a smooth slope joined to hanging particle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (4 kg) and B (8 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a smooth slope, $\alpha = 30°$, and B is 2.75 m above the ground.
Released from rest, find the further distance A moves up the slope after B lands (A does not reach the pulley).

A: $T - 4 \times 9.8 \times \frac{1}{2} = 4a$. B: $8 \times 9.8 - T = 8a$

Adding: $9.8(8 - 4 \times \frac{1}{2}) = 12a$, so $a = 4.9$

$T = 8(9.8 - 4.9) = 39.2$

When B lands: $v^2 = 2 \times 4.9 \times 2.75 = 26.95$

Then A decelerates at $g \sin \alpha = 4.9$: further distance $= \frac{26.95}{2 \times 4.9} = 2.75$

Answer: 2.75 m

2 Indices: simultaneous equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously, giving integer x and y:
$8^x \times 4^y = 2048$
$9^x \div 3^y = 243$

Write each side with prime bases: $2048 = 2^{11}$, $243 = 3^5$.
First: $2^{3x} \times 2^{2y} = 2^{11}$, so $3x + 2y = 11$.
Second: $3^{2x - 1y} = 3^5$, so $2x - 1y = 5$.
Solving the pair of linear equations: $x = 3$, $y = 1$.
Answer: x = 3, y = 1

3 Triple angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (triple angle))

Given that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, solve
$\cos 3\theta = -2\cos\theta$ for $0 < \theta < 2\pi$, in terms of π.

Substitute: $\cos\theta(4\cos^2\theta - 3 + 2) = 0$

$\cos\theta = 0$ or $\cos^2\theta = \frac{1}{4}$

$\cos\theta = 0$ gives $\frac{\pi}{2}, \frac{3\pi}{2}$

$\cos^2\theta = \frac{1}{4}$ gives $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Answer: $\theta = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

4 Cumulative distribution function (CIE)

The cdf of X is $F(x) = 0$ for $x < 0$, $F(x) = \frac{x^3}{125}$ for $0 \le x \le 5$, and $F(x) = 1$ for $x > 5$.

(a) Find the pdf of X.
(b) Find E(X).
(c) Find $P(1.25 < X < 2.5)$ and the median, to 3 significant figures.

(a) $f(x) = F'(x) = \frac{3}{125}x^2$ for $0 \le x \le 5$

(b) $E(X) = \int x\, f(x)\, dx$ from 0 to 5 $= \frac{15}{4}$

(c) $P = F(2.5) - F(1.25) = 0.1094$

Median m: $F(m) = 1/2$, so $m = 5 \times (1/2)^{1/3}$
$m = 3.969$

Answer: (a) $f(x) = \frac{3}{125}x^2$ (b) $\frac{15}{4}$ (c) 0.109, median 3.97

5 Volumes of revolution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The region bounded by $y = \frac{1}{\sqrt{x+3}}$, the x-axis and the lines $x = 0$ and $x = 6$ is rotated through 2π about the x-axis.
Find the exact volume of the solid formed.

$V = \pi\int_0^6 y^2\,dx = \pi\int_0^6 \left(\frac{1}{\sqrt{x+3}}\right)^2 dx$

Simplify the integrand, then integrate

$V = \pi \times (\ln 3)$

Answer: $\pi\ln 3$

6 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (2 kg) lies on a rough table with $\mu = 0.5$, and is joined by a light string over a smooth pulley (see diagram) to particle B (6 kg) on a smooth slope, $\alpha = 30°$.
Released from rest, the string is cut when B has moved 1.5 m. Find the further distance A slides before stopping.

A: $T - 0.5 \times 2 \times 9.8 = 2a$. B: $6 \times 9.8 \times \frac{1}{2} - T = 6a$

Adding: $9.8(6 \times \frac{1}{2} - 0.5 \times 2) = 8a$, so $a = 2.45$

$T = 2(2.45 + 0.5 \times 9.8) = 14.7$

When cut: $v^2 = 2 \times 2.45 \times 1.5 = 7.35$. Then A decelerates at $\mu g = 4.9$

Further distance $= \frac{7.35}{2 \times 4.9} = 0.75$

Answer: 0.75 m

7 Proof by contradiction (irrationality) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\sqrt{3}$ is irrational.

Assume $\sqrt{3} = \frac{a}{b}$ with a, b integers and no common factor.

Then $3b^2 = a^2$, so a^2 is a multiple of 3.
As 3 is prime, a is a multiple of 3: let $a = 3k$.
Then $3b^2 = 9k^2$, so $b^2 = 3k^2$, and b is a multiple of 3.
a and b have common factor 3: a contradiction. Hence $\sqrt{3}$ is irrational.

Answer: Proved

8 Equations in disguise (substitution) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$.

Let $u = x^{\frac{1}{3}}$: $u^2 - 5u + 6 = 0$.
$(u - 3)(u - 2) = 0$, so $u = 3$ or $u = 2$.
Cube: $x = 27$ or $x = 8$.

Answer: $x = 8$ or $x = 27$

9 Points of inflection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the coordinates of the points of inflection of

$y = \frac{4}{x^2 + 12}$.

$\frac{dy}{dx} = -\frac{8x}{(x^2 + 12)^2}$

$\frac{d^2y}{dx^2} = \frac{8(3x^2 - 12)}{(x^2 + 12)^3}$

Zero when $x^2 = 4$, so $x = \pm 2$, and the sign changes either side

$y = \frac{4}{16} = \frac{1}{4}$

Answer: $(2, \frac{1}{4})$ and $(-2, \frac{1}{4})$

10 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ab^x$. The line of best fit for lg y against x is lg y = 0.25 + 0.15x.
(a) Find a and b to 3 significant figures.
(b) Estimate y when x = 13, to 3 significant figures.
(c) Interpret the value of b.

$y = ab^x$ gives $\lg y = \lg a + x \lg b$
$\lg a = 0.25$ so $a = 10^{0.25} = 1.778$
$\lg b = 0.15$ so $b = 10^{0.15} = 1.413$
$x = 13$: $\lg y = 0.25 + 0.15 \times 13 = 2.2$
$y = 10^{2.2} = 158$
b is the factor y is multiplied by each time x increases by 1 (growth of 41%)
Answer: (a) a = 1.78, b = 1.41 (b) 158 (c) y grows by a factor of 1.41 per unit of x