Day 229 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Least force to drag a box (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 200 N is pulled at constant speed along rough horizontal ground ($\mu = \frac{7}{24}$) by a force P N at angle θ above the horizontal.

(a) Find P when $\tan\theta = \frac{4}{3}$. (b) Find the least possible value of P as θ varies, and the corresponding θ to 1 d.p.

Vertical: $R + P\sin\theta = 200$. Horizontal: $P\cos\theta = \mu R = \frac{7}{24}(200 - P\sin\theta)$

$P(\cos\theta + \frac{7}{24}\sin\theta) = \frac{7}{24} \times 200$, so $P = \frac{175}{3} \div (\cos\theta + \frac{7}{24}\sin\theta)$

(a) $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$: $P = \frac{175}{3} \div \frac{5}{6} = 70$

(b) P is least when $\cos\theta + \frac{7}{24}\sin\theta$ is greatest. Its greatest value is $\sqrt{(1 + \frac{7}{24}^2)} = \frac{25}{24}$,

which occurs when $\tan\theta = \frac{7}{24}$. Then $P = \frac{175}{3} \div \frac{25}{24} = 56$ and $\theta = 16.3°$

Answer: (a) 70 N (b) 56 N at θ = 16.3°

2 Identity then equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\frac{1 - \cos 2\theta}{1 + \cos 2\theta} \equiv \tan^2\theta$.

(b) Hence solve $\frac{1 - \cos 2\theta}{1 + \cos 2\theta} = 5$ for $0 \le \theta < 360°$ (1 d.p.).

$\frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \frac{2\sin^2\theta}{2\cos^2\theta} = \tan^2\theta$

$\tan^2\theta = 5$, so $\tan\theta = \pm\sqrt{5}$

$\theta = 65.9°$, then $180° - \theta$, $180° + \theta$, $360° - \theta$

Answer: θ = 65.9°, 114.1°, 245.9°, 294.1°

3 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = 4x^2 - 8x + 5$ for $x \ge k$. Given that k is the least value for which f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = 4(x - 1)^2 + 1$.

The vertex is at $x = 1$, so $k = 1$ (f is one-one on one side of it).

Let $y = f(x)$: $y - 1 = 4(x - 1)^2$, so $(x - 1) = \pm\sqrt{\frac{y - 1}{4}}$.

As $x \ge 1$, take the positive root: $x = 1 + \sqrt{\frac{y - 1}{4}}$.

Range of f = domain of f^{-1}: $f(1) = 1$, so $x \ge 1$.

Answer: $k = 1$; $f^{-1}(x) = 1 + \sqrt{\frac{x - 1}{4}}$; $x \ge 1$

4 Particle on a rough table, string slack when B lands (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (2 kg) and B (1 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a rough table with $\mu = 0.1$, and B is 1.5 m above the floor.

The system is released from rest. Find the total distance A moves before coming to rest (A does not reach the pulley).

Until B lands, A: $T - 0.1 \times 2 \times 9.8 = 2a$. B: $1 \times 9.8 - T = 1a$

Adding: $(1 - 0.1 \times 2) \times 9.8 = 3a$, so $a = \frac{196}{75}$

$v^2 = 2 \times \frac{196}{75} \times 1.5 = 7.84$, so $v = 2.8$

After B lands, A decelerates at $\mu g = 0.98$: extra distance $= \frac{2.8^2}{2 \times 0.98} = 4$

Total distance = 1.5 + 4

Answer: 5.5 m

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5 Large data set: linear change of units (AQA, Edexcel, OCR, MEI, Eduqas)

Large Data Set, daily total sunshine (hours), Leuchars, October 2015 (n = 31): mean 8.2, sd 2.3. (1 hour = 60 minutes)
(a) Find the mean and sd in minutes.
(b) Is a reading of 14 more than 2 sd above the mean?

New value = 60 × old, so mean = 60 × 8.2 = 492.0
The constant shifts the mean but not the spread; only the multiplier affects σ
σ = 60 × 2.3 = 138.0
Outlier cut-off = mean + 2σ = 8.2 + 4.6 = 12.8 (original units)
14 is greater than 12.8, so yes
Answer: (a) mean 492, sd 138 (b) yes

6 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation

$y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes $x = -2$ and $y = 2$, so $p = 2$ and $q = 2$.

$y = \frac{a}{x + 2} + 2$.

P(−4, 0): $0 = \frac{a}{-2} + 2$, so $a = 4$.

Answer: a = 4, p = 2, q = 2

7 Continuous random variable: pdf with unknown constant (CIE)

The pdf of X is $f(x) = kx(4 - x)$ for $0 \le x \le 4$, and 0 otherwise.
(a) Find k.
(b) Find E(X) and Var(X).
(c) Find P(X > 1).

(a) $\int kx(4 - x)\,dx$ from 0 to 4 $= k[2x^2 - x^3/3] = k \times \frac{32}{3} = 1$

$k = \frac{3}{32}$

(b) $E(X) = \int x f(x)\,dx = 2$ by symmetry about $x = 2$

$E(X^2) = \int x^2 f(x)\,dx = \frac{24}{5}$, so $Var(X) = \frac{24}{5} - 2^2 = \frac{4}{5}$

(c) $P(X > 1) = \int f(x)\,dx$ from 1 to 4 $= \frac{27}{32}$

Answer: (a) $k = \frac{3}{32}$ (b) E(X) = 2, Var(X) = $\frac{4}{5}$ (c) $\frac{27}{32}$

8 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$\frac{dy}{dx} = e^{x - y}$, and y = ln 2 when x = 0.

Show that $e^y = e^x + 1$ and hence find y in terms of x.

$e^y dy = e^x dx$, so $e^y = e^x + c$
x = 0: 2 = 1 + c, so c = 1
Take natural logarithms
Answer: $y = \ln(e^x + 1)$

9 Proof by contradiction (logarithms) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\log_{10} 7$ is irrational.

Assume $\log_{10} 7 = \frac{m}{n}$, m, n positive integers.

Then $10^{\frac{m}{n}} = 7$, so $10^m = 7^n$.

Prime factors of 10^m and 7^n must be identical,

but 7 divides 7^n but 7 does not divide 10^m.

This is a contradiction, so $\log_{10} 7$ is irrational.

Answer: Proved

10 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{4}{24 + 15 \sin\theta + 8 \cos\theta}$

(a) Write $15 \sin\theta + 8 \cos\theta$ as $R \sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{15^2 + 8^2} = 17$, $\tan\alpha = \frac{8}{15}$, $\alpha = 28.1°$

Denominator $= 24 + 17 \sin(\theta + 28.1°)$, which lies between 7 and 41

Greatest $f(\theta) = \frac{4}{7} = \frac{4}{7}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{4}{41} = \frac{4}{41}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 28.1° = 270°$, so $\theta = 241.9°$

Answer: (a) $17 \sin(\theta + 28.1°)$ (b) greatest $\frac{4}{7}$, least $\frac{4}{41}$, $\theta = 241.9°$