Day 228 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 6 m and mass 50 kg rests on supports at C and D, where AC = 2 m and DB = 2 m. A girl of mass 50 kg stands on it at distance x m from A.

Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $50(1) = 50(x - 4)$, so $x = 4 + \frac{50 \times 1}{50} = 5$

Tilting about C (girl before C): $50(1) = 50(2 - x)$, so $x = 2 - \frac{50 \times 1}{50} = 1$

Answer: $1 \leq x \leq 5$

2 Cubic graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the cubic $y = f(x)$, which touches the x-axis once. Find f(x), giving your answer in expanded form.

Crosses at $x = 2$ and touches at $x = -1$, so $f(x) = k(x - 2)(x + 1)^2$.

At $(0, -2)$: $-2 = k(-2)(1)^2 = -2k$, so $k = 1$.

$f(x) = (x - 2)(x + 1)^2$, then expand.

Answer: $f(x) = x^3 - 3x - 2$

3 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of

$y = \frac{e^x}{x^2}$ for $x > 0$,

and determine its nature.

$\frac{dy}{dx} = \frac{e^x(x - 2)}{x^3} = 0$ gives $x = 2$

$y = \frac{1}{4}e^2$ when $x = 2$

Second derivative at this point is positive, so it is a minimum

Answer: $(2, \frac{1}{4}e^2)$, minimum

4 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Evaluate $\sum_{r=4}^{12}(5 \times 3^r + 2r)$.

Geometric part: first term $5 \times 3^4 = 405$, ratio 3, 9 terms.

Sum $= \frac{405(3^9 - 1)}{2} = 3985605$.

Arithmetic part: 2 × 9 terms, average $(4 + 12)/2$: sum = 144.

Total = 3985605 + 144.

Answer: 3985749

5 Segments: exact area and perimeter (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A chord AB of a circle, centre O and radius 9 cm, subtends an angle of $\frac{2\pi}{3}$ at O.

Find the exact area and the exact perimeter of the minor segment cut off by AB.

Sector OAB: $\frac{1}{2}r^2\theta = \frac{1}{2} \times 9^2 \times \frac{2\pi}{3}$; triangle OAB: $\frac{1}{2}r^2\sin\theta = \frac{81}{4}\sqrt{3}$

Area $= 27\pi - \frac{81}{4}\sqrt{3}$

Chord AB $= 2r\sin(\theta/2) = 9\sqrt{3}$

Arc AB $= r\theta = 6\pi$

Perimeter = arc + chord

Answer: Area $= 27\pi - \frac{81}{4}\sqrt{3}$ cm^2, perimeter $= 6\pi + 9\sqrt{3}$ cm

6 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (4 kg) lies on a rough table with $\mu = 0.2$, and is joined by a light string over a smooth pulley (see diagram) to particle B (3 kg) on a smooth slope, $\alpha = 30°$.
Released from rest, the string is cut when B has moved 1.5 m. Find the further distance A slides before stopping.

A: $T - 0.2 \times 4 \times 9.8 = 4a$. B: $3 \times 9.8 \times \frac{1}{2} - T = 3a$

Adding: $9.8(3 \times \frac{1}{2} - 0.2 \times 4) = 7a$, so $a = 0.98$

$T = 4(0.98 + 0.2 \times 9.8) = 11.76$

When cut: $v^2 = 2 \times 0.98 \times 1.5 = 2.94$. Then A decelerates at $\mu g = 1.96$

Further distance $= \frac{2.94}{2 \times 1.96} = 0.75$

Answer: 0.75 m

7 Sum of two independent copies of a discrete variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X takes values 1, 2, 3, 4 with $P(X = x) = k(x + 1)$. X_1 and X_2 are independent copies of X.
(a) Find k.
(b) Find $P(X_1 + X_2 = 5)$.
(c) Find $P(X_1 = X_2)$ and $E(X_1 + X_2)$.

(a) $\Sigma P = k(2 + 3 + 4 + 5) = 14k = 1$, so $k = \frac{1}{14}$

(b) Ways to total 5: (1,4), (2,3), (3,2), (4,1)

$P = \frac{1}{7}\times\frac{5}{14} + \frac{3}{14}\times\frac{2}{7} + \frac{2}{7}\times\frac{3}{14} + \frac{5}{14}\times\frac{1}{7} = \frac{11}{49}$

(c) $P(X_1 = X_2) = \Sigma [P(X = x)]^2 = \frac{27}{98}$

$E(X) = \frac{20}{7}$, so $E(X_1 + X_2) = 2 \times \frac{20}{7} = \frac{40}{7}$

Answer: (a) $k = \frac{1}{14}$ (b) $\frac{11}{49}$ (c) $\frac{27}{98}$, $\frac{40}{7}$

8 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(26, \frac{1}{5})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{26 - r}{r + 1} \times \frac{1}{4}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(26, r + 1)}{C(26, r)} \times \frac{p}{q} = \frac{26 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{1}{5}$ and $q = \frac{4}{5}$, $p \div q = \frac{1}{4}$

The ratio exceeds 1 when $(26 - r)\frac{1}{5} > (r + 1)\frac{4}{5}$, that is $r < 4.4$

So probabilities rise up to r = 5 and then fall; the mode is 5
$P(X = 5) = 0.194$
Answer: 5 with probability 0.194

9 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A rectangle has its base on the x-axis and its upper vertices on the curve $y = 27 - x^2$, symmetric about the y-axis.
Find the maximum area of the rectangle, with justification.

Base 2x, height $27 - x^2$: $A = 2x(27 - x^2) = 54x - 2x^3$

$dA/dx = 54 - 6x^2 = 0$, so $x = 3$ $(x > 0)$

$d^2A/dx^2 = -12x < 0$: maximum

$A = 2 \times 3 \times 18$

Answer: 108

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10 Remainder theorem (quadratic divisor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the remainder when x^5 is divided by $(x - 1)(x + 1)$.

$x^5 = (x - 1)(x + 1)Q(x) + px + q$.
$x = 1$: $1 = p + q$.
$x = -1$: $-1 = -p + q$.
Subtract: $2p = 2$, so $p = 1$ and $q = 0$.
Answer: x