Day 227 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 8 m and mass 80 kg rests on supports at C and D, where AC = 3 m and DB = 3 m. A girl of mass 50 kg stands on it at distance x m from A.
Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $80(1) = 50(x - 5)$, so $x = 5 + \frac{80 \times 1}{50} = 6.6$

Tilting about C (girl before C): $80(1) = 50(3 - x)$, so $x = 3 - \frac{80 \times 1}{50} = 1.4$

Answer: $1.4 \leq x \leq 6.6$

2 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ax^n$. The line of best fit for lg y against lg x is $\lg y = 0.95 + 0.5 \lg x$.
(a) Find a and n, giving a to 3 significant figures.
(b) Estimate y when x = 5, to 3 significant figures.

$y = ax^n$ gives $\lg y = \lg a + n \lg x$, so the gradient is n and the intercept is lg a
$n = 0.5$
$\lg a = 0.95$ so $a = 10^{0.95} = 8.913$
$x = 5$: $y = 8.913 \times 5^{0.5}$
$= 19.9$
Answer: (a) a = 8.91, n = 0.5 (b) 19.9

3 Small angle approximations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (small angle))

θ is small and measured in radians. Use small angle approximations to find the value that $\frac{\sin 1\theta \tan 5\theta}{1 - \cos 5\theta}$ approaches as $\theta \to 0$.

$\sin 1\theta \approx 1\theta$, $\tan 5\theta \approx 5\theta$, $1 - \cos 5\theta \approx \frac{25}{2}\theta^2$

Expression $\approx \frac{1\theta \times 5\theta}{\frac{25}{2}\theta^2} = \frac{2}{5}$

Answer: $\frac{2}{5}$

4 Proof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Sam claims that $n^2 + 3n + 79$ is prime for every positive integer n.
Show that Sam is wrong by finding the smallest value of n that is a counter-example.

Test n = 1, 2, 3, ... in turn until the value is not prime.
For n = 5: $5^2 + 35 + 79 = 119$.
$119 = 7 \times 17$, which is not prime.
All smaller values of n give a prime, so n = 5 is the smallest.
Answer: n = 5 (value 119 = 7 × 17)

5 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

An arrow is projected from a point O on horizontal ground with speed 63.7 m s^{-1} at an angle α above the horizontal, where $\tan \alpha = \frac{12}{5}$. Find the horizontal distance between the two points at which the arrow is 171.5 m above the ground.
Take g = 9.8 m s^{-2}.

Vertical: $171.5 = 58.8\,t - 4.9t^2$, i.e. $4.9t^2 - 58.8t + 171.5 = 0$
$t = \frac{58.8 \pm \sqrt{(58.8^2 - 19.6 \times 171.5)}}{9.8} = \frac{58.8 \pm 9.8}{9.8}$
The two times differ by 2 s (symmetry about the top of the path)
Horizontal distance = 24.5 × 2
Answer: 49 m

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6 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ab^x$ passes through (0, 4) and (4, 1024).
Find the exact values of a and b, and the value of y when x = 6.

$ab^0 = 4$ and $ab^4 = 1024$
Divide: $b^4 = 256$ so $b = 4$
$a = 4 \div 4^0 = 4$
$y = 4 \times 4^6$
Answer: a = 4, b = 4, y = 16384

7 Perpendicular lines and area (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the diagram, line l_1 passes through A and B.
Line l_2 is perpendicular to l_1 at B and meets the
x-axis at C. Find the coordinates of C and the area of triangle ABC.

Gradient of $l_1 = \frac{9-3}{9} = \frac{2}{3}$.
Gradient of $l_2 = -\frac{3}{2}$. Equation: $y - 9 = -\frac{3}{2}x$.
y = 0: x = 6, so C(6, 0).
AB = $3\sqrt{13}$, BC = $3\sqrt{13}$: area = ½ × AB × BC.
Answer: C(6, 0); area $\frac{117}{2}$

8 Two-tailed binomial test (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A coin is said to show heads with probability 0.4. In 20 trials it does so 14 times.
Test at the 10% level whether the probability is different from 0.4.

H_0: p = 0.4, H_1: $p \neq 0.4$ (two-tailed). X ~ B(20, 0.4)
14 is above the mean 8, so use the upper tail: $P(X \geq 14) = 0.00647$
Each tail is compared with 5%: 0.00647 is less than 5%
Reject H_0: there is sufficient evidence that the probability is different from 0.4
Answer: Reject H_0 (p-value 0.00647 against 5%)

9 Equations in disguise (substitution) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 2 = 0$.

Let $u = x^{\frac{1}{3}}$: $u^2 - 3u + 2 = 0$.
$(u - 2)(u - 1) = 0$, so u = 2 or u = 1.
Cube: x = 8 or x = 1.
Answer: x = 1 or x = 8

10 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\frac{x^2 + 13x + 37}{(x+5)(x+4)} \equiv 1 + \frac{A}{x+5} + \frac{B}{x+4}$.

(b) Hence find the exact value of $\int_0^1 \frac{x^2 + 13x + 37}{(x+5)(x+4)}\,dx$.

Numerator = (x + 5)(x + 4) + A(x + 4) + B(x + 5)
x = −5 gives A = 3; x = −4 gives B = 1
Integral = [x + 3 ln(x + 5) + 1 ln(x + 4)]
Substitute limits 0 and 1
Answer: A = 3, B = 1; 1 + ln 2 + 3ln 3 − 2ln 5