**Day 226** **A Level Maths: S Tier 10-a-day: Model Solutions**

## 1 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ax^n$ passes through (1, 5) and (9, 15).
Find the values of a and n.

$5 = a \times 1^n$ and $15 = a \times 9^n$

Divide: $9^n = 3$ so $n = \frac{1}{2}$

$a = 5 \div 1^{\frac{1}{2}} = 5$

**Answer: a = 5, n = $\frac{1}{2}$**

## 2 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is dropped from a height of 5 m. After each bounce it rises to $\frac{3}{4}$ of its previous height.

(a) Find the total distance it travels before it stops.
(b) After how many bounces does it rise less than 0.1 m?

(a) Down H, then up and down each time: $H + 2Hr + 2Hr^2 + ...$

$$= H + \frac{2Hr}{1 - r} = 5 + \frac{2 \times 5 \times \frac{3}{4}}{1 - \frac{3}{4}} = 35.$$

(b) Rebound height after n bounces = $5(\frac{3}{4})^n < 0.1$.

$$n > \frac{\log(0.02)}{\log(0.75)} = 13.60, \text{ so } n = 14.$$

**Answer: (a) 35 m (b) 14 bounces**

## 3 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(25, \frac{2}{5})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{25 - r}{r + 1} \times \frac{2}{3}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$$P(X = r + 1) \div P(X = r) = \frac{C(25, r + 1)}{C(25, r)} \times \frac{p}{q} = \frac{25 - r}{r + 1} \times \frac{p}{q}$$

With $p = \frac{2}{5}$ and $q = \frac{3}{5}$, $p \div q = \frac{2}{3}$

The ratio exceeds 1 when $(25 - r)\frac{2}{5} > (r + 1)\frac{3}{5}$, that is $r < 9.4$

So probabilities rise up to r = 10 and then fall; the mode is 10
P(X = 10) = 0.161

**Answer: 10 with probability 0.161**

## 4 Geometric series: sum to infinity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has sum to infinity 8 and second term −6. Find the possible values of the common ratio r and the first term a in each case.

$a = \frac{-6}{r}$. $S_\infty = \frac{a}{1 - r} = \frac{-6}{r(1 - r)} = 8$.

$8r^2 - 8r - 6 = 0$.

$(2r + 1)(2r - 3) = 0$: $r = -\frac{1}{2}$ or $r = \frac{3}{2}$.

Need $|r| < 1$: $-\frac{1}{2}$ valid, $\frac{3}{2}$ rejected.

**Answer: r = $-\frac{1}{2}$, a = 12**

## 5 Triple angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (triple angle))

Given that $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$, solve
$\sin 3\theta = 2\sin\theta$ for $0 < \theta < 360°$.

Substitute: $\sin\theta(3 - 4\sin^2\theta - 2) = 0$

$\sin\theta = 0$ or $\sin^2\theta = \frac{1}{4}$

$\sin\theta = 0$ gives $180°$

$\sin^2\theta = \frac{1}{4}$ gives $30°, 150°, 210°, 330°$

**Answer: $\theta = 30°, 150°, 180°, 210°, 330°$**

## 6 Boat crossing a river (AQA, Edexcel, OCR, MEI)

A river is 80 m wide and flows at 10 m $s^{-1}$. A boat has speed 6 m $s^{-1}$ in still water and crosses the river heading so that it is carried as little distance downstream as possible.
Find the least possible distance downstream between the starting point and the landing point.

The boat speed is less than the current, so it cannot stop drifting. Let the boat head at angle $\varphi$ upstream from the perpendicular to the bank

Across: $6\cos\varphi$; along (downstream): $10 - 6\sin\varphi$. Drift = $\frac{80}{6\cos\varphi}(10 - 6\sin\varphi)$

Drift is least when the resultant velocity is at the greatest angle to the downstream direction: the boat velocity is perpendicular to the resultant, so $\sin\varphi = \frac{6}{10}$

Then across speed = $\frac{24}{5}$ and downstream speed = $\frac{32}{5}$, so drift = $80 \times \frac{\frac{32}{5}}{\frac{24}{5}}$

**Answer: $\frac{320}{3}$ m**

## 7 Waiting for the r-th success (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each shot Amira takes scores with probability $\frac{1}{3}$, independently.

(a) Find the probability that her fourth score comes on shot number 10.
(b) Find the probability that her fourth score comes on or before shot number 10.

(a) fourth score on shot 10: exactly 3 scores in the first 9 shots, then a score

$C(9, 3)(\frac{1}{3})^3(\frac{2}{3})^6 \times \frac{1}{3}$

$= 0.0910$

(b) On or before shot 10 means at least 4 scores in 10 shots: $X \sim B(10, \frac{1}{3})$

$P(X \geq 4) = 1 - P(X \leq 3) = 0.441$

**Answer: (a) 0.0910 (b) 0.441**

## 8 Sums of inverse tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B are acute angles with $\tan A = \frac{1}{7}$ and $\tan B = \frac{3}{4}$.

Find $\tan(A + B)$ and hence show that $A + B = \frac{\pi}{4}$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{3}{28}} = \frac{\frac{25}{28}}{\frac{25}{28}} = 1$$

$A + B$ is acute, so $A + B = \pi/4$

**Answer: $\tan(A + B) = 1$, $A + B = \frac{\pi}{4}$**

## 9 Rough slope: up then back down (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle is projected at 9 m $s^{-1}$ up a line of greatest slope of a rough plane inclined at $\alpha$ to the horizontal, where $\sin\alpha = \frac{3}{5}$ and $\mu = 0.35$.

Find (a) the distance it travels up the plane, to 3 s.f., (b) its speed on returning to its starting point. Take g = 9.8 m $s^{-2}$.

$\cos\alpha = \frac{4}{5}$. Up the plane: $R = mg\cos\alpha$, so $a_{up} = g(\sin\alpha + \mu\cos\alpha) = 9.8(\frac{3}{5} + 0.35 \times \frac{4}{5}) = 8.624$

$0 = 9^2 - 2 \times 8.624 \times d$, so $d = \frac{81}{17.248} = 4.696$

Down the plane: $a_{down} = g(\sin\alpha - \mu\cos\alpha) = 9.8(\frac{3}{5} - 0.35 \times \frac{4}{5}) = 3.136$

$v^2 = 2 \times 3.136 \times 4.6962 = 29.45$, so $v = 5.43$

**Answer: (a) 4.70 m (b) 5.43 m $s^{-1}$**

## 10 Proof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Sam claims that $n^2 + n + 17$ is prime for every positive integer n.
Show that Sam is wrong by finding the smallest value of n that is a counter-example.

Test n = 1, 2, 3, ... in turn until the value is not prime.

For n = 16: $16^2 + 16 + 17 = 289$.

289 = 17 × 17, which is not prime.

All smaller values of n give a prime, so n = 16 is the smallest.

**Answer: n = 16 (value 289 = 17 × 17)**