Day 224 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population of bacteria is modelled by $P = P_0e^{kt}$, t in hours.
Initially P = 500 and after 4 hours P = 625.
Find k, and the time for the population to reach 1000, to 3 s.f.

$P_0 = 500$. $500\,e^{4k} = 625$ so $e^{4k} = \frac{5}{4}$

$k = \frac{1}{4}\ln\frac{5}{4} = 0.0558$

$500\,e^{kt} = 1000$ so $e^{kt} = 2$, $t = \frac{1}{k}\ln 2$

t = 12.4 hours

Answer: k = 0.0558, t = 12.4 hours

2 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + 4x^2 + 12x + 27$.
Show that (x + 3) is a factor of f(x), and that the equation f(x) = 0 has exactly one real root.

$f(-3) = -27 + 36 - 36 + 27 = 0$, so (x + 3) is a factor.

$f(x) = (x + 3)(x^2 + x + 9)$.

For $x^2 + x + 9 = 0$: $b^2 - 4ac = 1 - 36 = -35 < 0$.

No real roots from the quadratic, so the only real root is $x = -3$.

Answer: Proved; the only real root is $x = -3$

3 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

An arrow is projected from a point O on horizontal ground with speed 49 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{4}{3}$. Find (a) the greatest height, (b) the speed when it is at half of this height (3 s.f.). Take g = 9.8 m s^{-2}.

Vertical: $0 = 39.2^2 - 2 \times 9.8 \times H$, so H = 78.4

At height $\frac{H}{2} = 39.2$: $v_y^2 = 39.2^2 - 2 \times 9.8 \times 39.2 = 768.32$

$v_x = 29.4$ (constant), so $\text{speed}^2 = 864.36 + 768.32 = 1632.68$

Answer: (a) 78.4 m (b) 40.4 m s^{-1}

4 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation $y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes x = 3 and y = 1, so p = −3 and q = 1.

$y = \frac{a}{x - 3} + 1$.

P(4, −1): $-1 = \frac{a}{1} + 1$, so a = −2.

Answer: a = −2, p = −3, q = 1

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5 Constant speed up a slope, force at an angle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 170 N is pulled at constant speed up a line of greatest slope of a rough plane ($\tan\alpha = \frac{3}{4}$, $\mu = \frac{7}{24}$) by a force P N at angle β above the plane.

(a) Find P when $\tan\beta = \frac{4}{3}$. (b) Find the least possible value of P as β varies.

Perpendicular to the plane: $R + P\sin\beta = 170\cos\alpha$, so $R = 170 \times \frac{4}{5} - P\sin\beta$

Along the plane: $P\cos\beta = 170\sin\alpha + \mu R$, so $P(\cos\beta + \mu\sin\beta) = 170(\sin\alpha + \mu\cos\alpha)$

(a) $P(\frac{3}{5} + \frac{7}{24} \times \frac{4}{5}) = 170(\frac{3}{5} + \frac{7}{24} \times \frac{4}{5})$, so $P = 170$

(b) P is least when $\cos\beta + \mu\sin\beta$ is greatest, i.e. $\sqrt{(1+\mu^2)} = \frac{25}{24}$ (when $\tan\beta = \frac{7}{24}$)

$P_{min} = 170(\frac{3}{5} + \frac{7}{24} \times \frac{4}{5}) \div \frac{25}{24}$

Answer: (a) 170 N (b) 136 N

6 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 2x^3 + ax^2 + bx - 36$, where a and b are constants.
$(x + 3)$ is a factor of f(x), and f(x) leaves remainder 0 when divided by $(x + 2)$.
Find a and b, and hence solve $f(x) = 0$.

$f(-3) = 0$: $-54 + 9a - 3b - 36 = 0$, so $9a - 3b - 90 = 0$.
$f(-2) = 0$: $4a - 2b - 52 = 0$.
Solving: $a = 4$, $b = -18$.
$f(x) = 2x^3 + 4x^2 - 18x - 36 = 2(x + 3)(x + 2)(x - 3)$

Answer: a = 4, b = −18; x = −3, −2, 3

7 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Write $\frac{-x-7}{(x+5)^2}$ in the form $\frac{A}{x+5} + \frac{B}{(x+5)^2}$.

(b) Find the exact value of $\int_0^1 \frac{-x-7}{(x+5)^2}\,dx$.

$-x - 7 \equiv A(x + 5) + B$, so $A = -1$, and $B = -2$

Integral $= [-1\ln(x + 5) - \frac{-2}{x+5}]$

Substitute limits 0 and 1

Answer: (a) A = −1, B = −2; (b) $-\frac{1}{15} - \ln 2 - \ln 3 + \ln 5$

8 Waiting for the r-th success (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each shot Amira takes scores with probability $\frac{1}{4}$, independently.

(a) Find the probability that her third score comes on shot number 5.
(b) Find the probability that her third score comes on or before shot number 5.

(a) third score on shot 5: exactly 2 scores in the first 4 shots, then a score

$C(4, 2)(\frac{1}{4})^2(\frac{3}{4})^2 \times \frac{1}{4}$

$= 0.0527$

(b) On or before shot 5 means at least 3 scores in 5 shots: $X \sim B(5, \frac{1}{4})$

$P(X \geq 3) = 1 - P(X \leq 2) = 0.104$

Answer: (a) 0.0527 (b) 0.104

9 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (3, −2, 4), (1, 1, −2), (0, −2, 4).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$\text{BA} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$, $\text{BC} = \begin{pmatrix} -1 \\ -3 \\ 6 \end{pmatrix}$; $\text{BA} \cdot \text{BC} = 43$

$|\text{BA}| = \sqrt{49}$, $|\text{BC}| = \sqrt{46}$

$\cos \text{ABC} = \frac{43}{\sqrt{49}\sqrt{46}} = 0.9057$, so $\text{ABC} = 25.1°$

$|\text{BA} \times \text{BC}| = \left|\begin{pmatrix} 0 \\ -18 \\ -9 \end{pmatrix}\right| = \sqrt{405}$

$\text{Area} = \frac{1}{2}|\text{BA} \times \text{BC}|$

Answer: ABC = 25.1°, area = $\frac{9}{2}\sqrt{5}$

10 Poisson: scaling the rate, then binomial (CIE)

Calls arrive at a switchboard at an average rate of 4 per hour, independently and at random.
(a) Find the probability of at least 3 in a 10-minute period.
(b) Find the probability that, of 6 separate 10-minute periods, exactly 2 have no arrivals.

10-minute period: $\lambda = 4 \times \frac{10}{60} = \frac{2}{3}$, so $X \sim \text{Po}(\frac{2}{3})$

(a) $P(X \geq 3) = 1 - P(X \leq 2) = 0.03021$

(b) $P(X = 0) = e^{-\frac{2}{3}} = 0.5134$
$Y \sim B(6, 0.5134)$: $P(Y = 2) = 0.222$
Answer: (a) 0.0302 (b) 0.222