Day 221 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A radioactive sample has mass $m = m_0 e^{-kt}$ grams, t in days.
Initially m = 100 and after 3 days m = 50.
Find k, and the time for the mass to fall to 5 grams, to 3 s.f.

$m_0 = 100$. $100\,e^{-3k} = 50$ so $e^{-3k} = \frac{1}{2}$

$k = \frac{1}{3} \ln 2 = 0.231$

$100\,e^{-kt} = 5$ so $e^{kt} = 20$, $t = \frac{1}{k} \ln 20$

t = 13.0 days

Answer: k = 0.231, t = 13.0 days

2 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each seed is faulty with probability 0.02, independently.
Find the least number n of seeds that must be tested so that the probability of finding at least one faulty is greater than 0.99.

P(at least one faulty) $= 1 - 0.98^n > 0.99$
$0.98^n < 0.01$
$n \ln 0.98 < \ln 0.01$
Dividing by ln 0.98 (negative) reverses the inequality: $n > 227.9$
n = 228

Answer: n = 228

3 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 11 seats. Each row has 3 more seats than the row in front. Find
(a) the number of seats in row 58,
(b) the least number of rows for at least 1094 seats.

Arithmetic: a = 11, d = 3.
(a) $u_{58} = 11 + 57 \times 3 = 182$.
(b) $S_n = \frac{n}{2}(22 + 3n - 3) = \frac{n(3n + 19)}{2} \geq 1094$.
$S_{24} = 1092 < 1094$, $S_{25} = 1175 \geq 1094$.

Answer: (a) 182 (b) 25 rows

4 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground with speed 63.7 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{12}{5}$. Find (a) the time of flight, (b) the range, (c) the greatest height. Take g = 9.8 m s^{-2}.

Components: horizontal 24.5, vertical 58.8 (as $\sin\theta = \frac{12}{13}$, $\cos\theta = \frac{5}{13}$)

Time of flight: $0 = 58.8\,t - 4.9t^2$, so $t = \frac{58.8}{4.9} = 12$

Range $= 24.5 \times 12 = 294$

Greatest height: $0 = 58.8^2 - 2 \times 9.8 \times H$, so $H = \frac{3457.44}{19.6} = 176.4$

Answer: (a) 12 s (b) 294 m (c) 176.4 m

5 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $8 \sin\theta + 15 \cos\theta$ in the form $R \sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $8 \sin\theta + 15 \cos\theta = 7$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{8^2 + 15^2} = 17$ and $\tan\alpha = \frac{15}{8}$, so $\alpha = 61.9°$

So $8 \sin\theta + 15 \cos\theta \equiv 17 \sin(\theta + 61.9°)$

$\sin(\theta + \alpha) = \frac{7}{17}$

$\theta + \alpha = 24.3°$ or $155.7°$ (using $\alpha = 61.9°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) R = 17, α = 61.9° (b) θ = 93.8°, 322.4°

6 Arithmetic series from a sum formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The sum of the first n terms of a sequence is
$S_n = 3n^2 + 7n$.
(a) Find an expression for u_n.
(b) Find the least n for which $u_n > 316$.

$u_n = S_n - S_{n-1} = 3n^2 + 7n - [3(n-1)^2 + 7(n-1)]$

$= 6n + 4$.

$u_1 = S_1 = 10$ also fits, so $u_n = 6n + 4$ for all n.

$6n + 4 > 316 \Rightarrow n > 52$, so n = 53.

Answer: (a) $u_n = 6n + 4$ (b) n = 53

7 Shaded normal curve and the standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The pdf of $X \sim N(32, \sigma^2)$ is shown; the shaded area is symmetrical.
(a) Find σ. (b) Find $P(X > 51)$.

The area 0.9 leaves 0.100 in total in the tails, 0.050 in each

$P(X < 24) = 0.050$ so $\frac{24 - 32}{\sigma} = -1.6449$

$\sigma = \frac{8}{1.6449} = 4.864$

$P(X > 51) = P(Z > 3.907) = 0.0000468$

Answer: (a) 4.86 (b) 0.0000468

8 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two drones P and Q move so that, t seconds after launch, their position vectors in metres are
$\mathbf{r}_P = \begin{pmatrix} -2 \\ -5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix} -2 \\ -6 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$.
Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0$

$-3 + t(1) = 0$, so t = 3

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, with length $\sqrt{1}$

Answer: t = 3, minimum distance = 1 m

9 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:

$x - y = 6$

$x^2 - xy - y^2 = 9$

From the first equation $x = 6 + y$.

Substitute into the second equation and collect terms:

this gives $-y^2 + 6y + 27 = 0$.

$y = -3$ or $y = 9$, giving $x = 3$ or $x = 15$.

Answer: (x, y) = (3, −3) and (15, 9)

10 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(-2\mathbf{i} + 2\mathbf{j})$ N, $(3\mathbf{i} + k\mathbf{j})$ N and $(3\mathbf{i} + 3\mathbf{j})$ N act on a particle of mass 2 kg. The resultant is parallel to $(4\mathbf{i} + \mathbf{j})$. Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(4)\mathbf{i} + (5 + k)\mathbf{j}$

Parallel to $(4\mathbf{i} + \mathbf{j})$: $\frac{5 + k}{4} = \frac{1}{4}$, so $k = -4$

$\mathbf{R} = (4\mathbf{i} + \mathbf{j})$ N, so $|R| = \sqrt{(4^2 + 1^2)} = 4.123$

$a = |R| \div 2$

Answer: (a) k = −4 (b) 2.06 m s^{-2}