Day 22 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx - 4}{x - 5}$ for $x \neq 5$. Given that f is self-inverse, find k, and hence solve $f(x) = x$.

Let $y = f(x)$: $y(x - 5) = kx - 4$, so $x(y - k) = 5y - 4$.

$f^{-1}(x) = \frac{5x - 4}{x - k}$.

$f = f^{-1}$ requires $k = 5$.

$f(x) = x$: $5x - 4 = x(x - 5)$, so $x^2 - 10x + 4 = 0$.

$x = 5 \pm \sqrt{21} = 5 \pm \sqrt{21}$.

Answer: $k = 5$; $x = 5 \pm \sqrt{21}$

2 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 16 and variance $\frac{16}{3}$.

(a) Find n and p.
(b) Find P(X = 18), correct to 3 significant figures.

$np = 16$ and $np(1 - p) = \frac{16}{3}$

Divide: $1 - p = \frac{16}{3} \div 16 = \frac{1}{3}$, so $p = \frac{2}{3}$

$n = 16 \div \frac{2}{3} = 24$

$P(X = 18) = C(24, 18) (\frac{2}{3})^{18} (\frac{1}{3})^{6} = 0.125$

Answer: (a) $n = 24$, $p = \frac{2}{3}$ (b) 0.125

3 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A stone is projected from a point O on horizontal ground with speed 49 m s^{-1} at an angle α above the horizontal, where $\tan \alpha = \frac{4}{3}$. Find the horizontal distance between the two points at which the stone is 73.5 m above the ground. Take $g = 9.8$ m s^{-2}.

Vertical: $73.5 = 39.2\,t - 4.9t^2$, i.e. $4.9t^2 - 39.2t + 73.5 = 0$

$t = \frac{39.2 \pm \sqrt{(39.2^2 - 19.6 \times 73.5)}}{9.8} = \frac{39.2 \pm 9.8}{9.8}$

The two times differ by 2 s (symmetry about the top of the path)
Horizontal distance = 29.4×2
Answer: 58.8 m

4 Conditions on the roots of a quadratic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

One root of $x^2 + kx + 64 = 0$ is the square of the other root. Find the possible values of k.

Let the roots be α and α^2.
Product: $\alpha^3 = 64$, so $\alpha = 4$ (the only real cube root).
Sum: $\alpha + \alpha^2 = 4 + 16 = 20 = -k$.
Answer: $k = -20$

5 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $3\sin\theta + 4\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $3\sin\theta + 4\cos\theta = 3$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{3^2 + 4^2} = 5$ and $\tan\alpha = \frac{4}{3}$, so $\alpha = 53.1°$

So $3\sin\theta + 4\cos\theta \equiv 5\sin(\theta + 53.1°)$

$\sin(\theta + \alpha) = \frac{3}{5}$

$\theta + \alpha = 36.9°$ or $143.1°$ (using $\alpha = 53.1°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) R = 5, α = 53.1° (b) θ = 90.0°, 343.7°

6 Just clearing a wall at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 45° to the horizontal. The ball just passes over the top of a vertical wall of height 8.8 m, at a horizontal distance of 48 m from O.
Find (a) the speed of projection, (b) the greatest height of the ball above the ground, to 3 s.f. Take $g = 9.8$ m s^{-2}.

At 45°: $x = \frac{u\,t}{\sqrt{2}}$ and $y = \frac{u\,t}{\sqrt{2}} - 4.9t^2$, so $y = x - \frac{9.8x^2}{u^2}$

$8.8 = 48 - \frac{9.8 \times 48^2}{u^2}$, so $u^2 = \frac{9.8 \times 2304}{39.2} = 576$

$u = 24$

Greatest height $= \frac{(u\sin 45°)^2}{(2 \times 9.8)} = \frac{576}{39.2}$

Answer: (a) 24 m s^{-1} (b) 14.7 m

7 Stationary points of x^n times a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = x\sqrt{27 - x}$, for $0 < x < 27$, has one stationary point. Find its exact coordinates and determine its nature.

$\frac{dy}{dx} = \frac{(54 - 3x)}{2\sqrt{27 - x}}$

$\frac{dy}{dx} = 0$: $54 - 3x = 0$, so $x = 18$

$y = 18 \times 3$

Gradient changes from positive to negative: a maximum

Answer: (18, 54), maximum

8 Cubic graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the cubic $y = f(x)$, which touches the x-axis once. Find $f(x)$, giving your answer in expanded form.

Crosses at $x = -3$ and touches at $x = -2$, so $f(x) = k(x + 3)(x + 2)^2$.

At (0, 24): $24 = k(3)(2)^2 = 12k$, so $k = 2$.

$f(x) = 2(x + 3)(x + 2)^2$, then expand.

Answer: f(x) = 2x^3 + 14x^2 + 32x + 24

9 Points of inflection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the point of inflection of
$y = (x + 2)e^{-x}$,
justifying that it is a point of inflection.

$\frac{dy}{dx} = e^{-x}(1 - (x + 2))$

$\frac{d^2y}{dx^2} = e^{-x}((x + 2) - 2) = 0$, so $x = 0$

$\frac{d^2y}{dx^2}$ changes sign either side of $x = 0$

$y = 2$

Answer: (0, 2)

10 Tree diagram with an unknown probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tree diagram shows two games played in turn.

Given the second is a W, P(first was a W) = $\frac{1}{2}$. Find x.

P(W then W) = 0.8x, P(second W) = 0.8x + 0.2(1 − x)

P(first W | second W) = $\frac{0.8x}{0.8x + 0.2(1 - x)} = \frac{1}{2}$

$0.8x = \frac{1}{2}(0.8x + 0.2 - 0.2x)$

Solve the linear equation for x: $x = \frac{1}{5}$

Answer: $x = \frac{1}{5}$