Day 218 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{1+2\sqrt{2}}{4+\sqrt{2}}$ in the form $p + q\sqrt{2}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(4 - \sqrt{2})$.

Numerator: $(1 + 2\sqrt{2})(4 - \sqrt{2}) = 7\sqrt{2}$.

Denominator: $4^2 - 1^2(2) = 14$.

So the fraction = $\frac{7\sqrt{2}}{14}$.

Answer: $\mathbf{\frac{1}{2}\sqrt{2}}$

2 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 27 and variance 6.75.

(a) Find n and p.

(b) Find P(X = 26), correct to 3 significant figures.

$np = 27$ and $np(1 - p) = 6.75$

Divide: $1 - p = 6.75 \div 27 = \frac{1}{4}$, so $p = \frac{3}{4}$

$n = 27 \div \frac{3}{4} = 36$

$P(X = 26) = C(36, 26)\ (\frac{3}{4})^{26}\ (\frac{1}{4})^{10} = 0.137$

Answer: (a) n = 36, p = $\frac{3}{4}$ (b) 0.137

3 Identity then equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\frac{1-\cos 2\theta}{1+\cos 2\theta} \equiv \tan^2\theta$.

(b) Hence solve $\frac{1-\cos 2\theta}{1+\cos 2\theta} = \frac{3}{2}$ for $0 \le \theta < 360°$ (1 d.p.).

$\frac{1-\cos 2\theta}{1+\cos 2\theta} = \frac{2\sin^2\theta}{2\cos^2\theta} = \tan^2\theta$

$\tan^2\theta = \frac{3}{2}$, so $\tan\theta = \pm\sqrt{\frac{3}{2}}$

$\theta = 50.8°$, then $180° - \theta$, $180° + \theta$, $360° - \theta$

Answer: θ = 50.8°, 129.2°, 230.8°, 309.2°

4 Binomial expansion (fractional or negative powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the binomial expansion of $(4 + 6x)^{-\frac{1}{2}}$ in ascending powers of x up to and including the term in x^3.

State the values of x for which it is valid.

$(4 + 6x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}(1 + \frac{3}{2}x)^{-\frac{1}{2}} = \frac{1}{2}(1 + u)^{-\frac{1}{2}}$, $u = \frac{3}{2}x$.

$(1 + u)^{-\frac{1}{2}} = 1 + (-\frac{1}{2})u + \frac{3}{8}u^2 + (-\frac{5}{16})u^3 + ...$

Substitute u and multiply by $\frac{1}{2}$.

Valid when $|u| < 1$, i.e. $|x| < \frac{2}{3}$.

Answer: $\frac{1}{2} - \frac{3}{8}x + \frac{27}{64}x^2 - \frac{135}{256}x^3 + ...$; $|x| < \frac{2}{3}$

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5 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass M kg, carrying a passenger of mass 60 kg, hangs from a vertical cable. The tension in the cable is 5676 N and the floor exerts a force of 516 N on the passenger.

Find (a) the acceleration of the lift, with its direction, (b) the value of M. Take g = 9.8 m s^{-2}.

Passenger (upwards positive): $516 - 60 \times 9.8 = 60\,a$, so $a = -1.2$

$a = 1.2$ m s^{-2}, so the acceleration is downwards

Lift and passenger: $5676 - (M + 60) \times 9.8 = (M + 60) \times -1.2$

$(M + 60)(9.8 - 1.2) = 5676$, so $M + 60 = \frac{5676}{8.6} = 660$

Answer: (a) 1.2 m s^{-2} downwards (b) M = 600

6 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each bulb is faulty with probability 0.08, independently.

Find the least number n of bulbs that must be tested so that the probability of finding at least one faulty is greater than 0.995.

P(at least one faulty) $= 1 - 0.92^n > 0.995$

$0.92^n < 0.005$

$n \ln 0.92 < \ln 0.005$

Dividing by ln 0.92 (negative) reverses the inequality: $n > 63.54$

$n = 64$

Answer: n = 64

7 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curves $y = e^x$ and $y = 9\,e^{-x}$ meet at the point P.

Find the exact area of the region bounded by the two curves and the y-axis.

$e^x = 9\,e^{-x}$ gives $e^{2x} = 9$, so $e^x = 3$, $x = \ln 3$

Area $= \int_0^{\ln 3} (9\,e^{-x} - e^x)\,dx = [-9\,e^{-x} - e^x]$

$= (-3 - 3) - (-9 - 1)$

Answer: 4

8 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$\frac{dy}{dx} = e^{x-y}$, and $y = \ln 6$ when $x = 0$.

Show that $e^y = e^x + 5$ and hence find y in terms of x.

$e^y dy = e^x dx$, so $e^y = e^x + c$

$x = 0$: $6 = 1 + c$, so $c = 5$

Take natural logarithms

Answer: y = ln(e^x + 5)

9 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations

$x = 3t$, $y = 4t^2$. The line $y = -\frac{28}{3}x - 48$

meets C at A and B. Find the exact length AB.

Substitute: $4t^2 = -28t - 48$.

$4t^2 + 28t + 48 = 0 \Rightarrow t = -3$ or $t = -4$.

A(−9, 36), B(−12, 64).

$AB^2 = 3^2 + (-28)^2 = 793$.

Answer: AB = $\sqrt{793}$

10 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time t = 0 hours, A is at $(-5\mathbf{i} - 4\mathbf{j})$ km relative to a fixed origin and B is at $(-3\mathbf{i} - 15\mathbf{j})$ km. A has velocity $(5\mathbf{i} + 5\mathbf{j})$ km h^{-1} and B has velocity $(2\mathbf{i} + 9\mathbf{j})$ km h^{-1}.
Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (2\mathbf{i} - 11\mathbf{j}) + t(-3\mathbf{i} + 4\mathbf{j})$

$|\mathbf{d}|^2 = (-3t + 2)^2 + (4t - 11)^2 = 25t^2 - 100t + 125$

Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\frac{-50}{25} = 2$

$\mathbf{d} = (2\mathbf{i} - 11\mathbf{j}) + 2(-3\mathbf{i} + 4\mathbf{j}) = (-4\mathbf{i} - 3\mathbf{j})$, so least distance = 5

Answer: (a) t = 2 h (b) 5 km