Day 216 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 2x^3 + ax^2 + bx + 32$, where a and b are constants.
$(x - 4)$ is a factor of f(x), and f(x) leaves remainder 56 when divided by $(x + 3)$.
Find a and b, and hence solve $f(x) = 0$.

$f(4) = 0$: $128 + 16a + 4b + 32 = 0$, so $16a + 4b + 160 = 0$.
$f(-3) = 56$: $9a - 3b - 22 = 56$.
Solving: $a = -2$, $b = -32$.
$f(x) = 2x^3 - 2x^2 - 32x + 32 = 2(x + 4)(x - 1)(x - 4)$
Answer: a = −2, b = −32; x = −4, 1, 4

2 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time t = 0 hours, A is at $(3\mathbf{i} + \mathbf{j})$ km relative to a fixed origin and B is at $(-2\mathbf{i} + 11\mathbf{j})$ km. A has velocity $(-5\mathbf{i} + 5\mathbf{j})$ km h^{-1} and B has velocity $(-\mathbf{i} + 2\mathbf{j})$ km h^{-1}.
Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (-5\mathbf{i} + 10\mathbf{j}) + t(4\mathbf{i} - 3\mathbf{j})$
$|\mathbf{d}|^2 = (4t - 5)^2 + (-3t + 10)^2 = 25t^2 - 100t + 125$
Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\frac{-50}{25} = 2$
$\mathbf{d} = (-5\mathbf{i} + 10\mathbf{j}) + 2(4\mathbf{i} - 3\mathbf{j}) = (3\mathbf{i} + 4\mathbf{j})$, so least distance = 5
Answer: (a) t = 2 h (b) 5 km

3 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $21 \sin\theta + 20 \cos\theta$ in the form $R \sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $21 \sin\theta + 20 \cos\theta = -7$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{21^2 + 20^2} = 29$ and $\tan\alpha = \frac{20}{21}$, so $\alpha = 43.6°$
So $21 \sin\theta + 20 \cos\theta \equiv 29 \sin(\theta + 43.6°)$
$\sin(\theta + \alpha) = \frac{-7}{29}$
$\theta + \alpha = -14.0°$ or $194.0°$ (using $\alpha = 43.6°$)
Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$
Answer: (a) R = 29, α = 43.6° (b) θ = 150.4°, 302.4°

4 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx - 7}{x - 3}$ for $x \neq 3$. Given that f is self-inverse, find k, and hence solve $f(x) = x$.

Let $y = f(x)$: $y(x - 3) = kx - 7$, so $x(y - k) = 3y - 7$.
$f^{-1}(x) = \frac{3x - 7}{x - k}$.
$f = f^{-1}$ requires $k = 3$.
$f(x) = x$: $3x - 7 = x(x - 3)$, so $x^2 - 6x + 7 = 0$.
$x = 3 \pm \sqrt{2} = 3 \pm \sqrt{2}$.
Answer: k = 3; x = 3 ± √2

5 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 2 yellow counters and some white counters, n in total.

Two are taken without replacement. P(both yellow) = $\frac{1}{78}$. Find n.

P(both yellow) = $\frac{2}{n} \times \frac{1}{(n-1)} = \frac{2}{n(n-1)}$
So $n(n - 1) = 2 \div \frac{1}{78} = 156$
$n^2 - n - 156 = 0$, so $(n - 13)(n + 12) = 0$
$n > 0$, so $n = 13$
Answer: n = 13

6 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = \ln t$, $y = 5t^2 - 4t$, $t > 0$.
Find the equation of the tangent where $t = 1$, in the form $y = mx + c$.

$dx/dt = \frac{1}{t}$, $dy/dt = 10t - 4$

$dy/dx = t(10t - 4)$ which is 6 at $t = 1$
At $t = 1$: $x = 0$, $y = 1$
Answer: $y = 6x + 1$

7 Perpendicular lines and area (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the diagram, line l_1 passes through A and B.
Line l_2 is perpendicular to l_1 at B and meets the
x-axis at C. Find the coordinates of C and the area of triangle ABC.

Gradient of $l_1 = \frac{3 - 1}{-1 + 3} = 1$.

Gradient of $l_2 = -1$. Equation: $y - 3 = -1(x + 1)$.
$y = 0$: $x = 2$, so C(2, 0).
AB = $2\sqrt{2}$, BC = $3\sqrt{2}$: area = ½ × AB × BC.
Answer: C(2, 0); area 6

8 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A closed cylinder has volume 128π cm^3. Find, in terms of π, the
minimum possible total surface area, proving that it is a minimum.

$h = \frac{128}{r^2}$, $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{256\pi}{r}$

$dS/dr = 4\pi r - \frac{256\pi}{r^2} = 0$, so $r^3 = 64$, $r = 4$

$d^2S/dr^2 = 4\pi + \frac{512\pi}{r^3} > 0$, so a minimum

$S = 2\pi \times 16 + \frac{256\pi}{4}$

Answer: 96π cm^2

9 Normal test: critical region and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim N(\mu, 100)$. H_0: $\mu = 89$ is tested against H_1: $\mu > 89$ using the mean of a sample of size 9 at the 1% level.
(a) Find the critical region for the sample mean.
(b) Find the probability of a Type II error when $\mu = 94$.

Under H_0, sample mean ~ $N(89, \frac{100}{9})$, standard error = 3.333

$z = 2.3263$ for the upper tail: critical value = $89 + 2.3263 \times 3.333 = 96.75$
Type II error: do not reject H_0, sample mean < 96.75, where mean ~ N(94, 3.333)

$z = \frac{96.75 - 94}{3.333} = 0.826$

Probability = 0.796
Answer: (a) sample mean > 96.8 (b) 0.796

10 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 5 kg lies on a smooth horizontal table, with a block of mass 3 kg on it ($\mu = 0.25$ between them). A horizontal force P N acts on the plank.

(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 24.5. Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{8}$. The block is accelerated only by friction: $F = 3a \leq \mu mg$

So $a \leq \mu g = 2.45$, and $P_{max} = 8 \times 2.45 = 19.6$

$P = 24.5 > 19.6$, so the block slips. Block: $a = \mu g = 2.45$

Plank: $24.5 - 7.35 = 5a$, so $a = 3.43$

Answer: (a) 19.6 N (b) block 2.45 m s^{-2}, plank 3.43 m s^{-2}